Quantum Field Theory in Curved Space-Time

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Email corrections and queries to the address above.

QFT in Curved Space-Time

Number of lectures: 16 TT2018
No formal assessment; homework completion requirement.
Weight: One unit
Areas: PT, Astro.
Prequels/pre-requisites: Quantum Field Theory (MT), General Relativity I (MT).

Overview

This course builds on both the first courses in quantum field theory and general relativity. The second course in GR and a course on differential geometry will be helpful, but are not essential.

Learning Outcomes

Students will be able to formulate classical and quantum field theories in curved space-time including an understanding of global features.

Syllabus

Non-interacting fields in curved space-time: Lagrangians, coupling to gravity, spinors in curved space-time, global hyperbolicity, asymptotic structure, conformal properties. Black hole thermodynamics. Canonical formulation.

Quantization, choice of vacuum. Quantum fields in Anti de Sitter space. Quantum fields in an expanding universe. Unruh effect. Casimir effect. Hawking radiation. Interacting quantum fields in curved space-time.

Reading List

There are many texts. The section on global structure, spinors and classical field theory on curved space-time partly follows the following books;

Hawking & Ellis, The large scale structure of Space-time, 1971 CUP.

Penrose & Rindler, Spinors & Space-time, Vols 1 & 2, CUP, 1984 & 1986.

Those that go further into the QFT include:

R Wald, QFT in Curved Space-time and Black Hole Thermodynamics, Univ Chicago Press, 1994, ISBN 0226-87027-8.

Birrell & Davis, Quantum field theory in curved space-time, CUP.

Ford, Quantum Field theory in Curved space-time, arxiv:9707062.

Gibbons/Hawking/Townsend, Black Holes lecture notes, arxiv:9707012.

Jacobson, Introduction to quantum fields in curved space-time and the Hawking effect, arxiv:0308048.

Mukhanov and Winitzki, Introduction to quantum fields on classical backgrounds.

1 Introduction

The goal of this course is free, i.e., non-interacting, classical and quantum fields in curved space-time. This is a first essential step towards interacting quantum field theory on a curved background, and beyond to quantum gravity. Already, there are two main areas of application

- Black hole thermodynamics: Hawking radiation provides the temperature in Bekenstein's analogies between properties of black holes and thermodynamics, with the area playing role of entropy.

- In cosmology, the cosmic microwave background spectrum is one naive consequence of QFT in curved space-time. The fluctuations that caused the creation of galaxies are also thought to have a quantum origin.

More recently these ideas have played a role in AdS/CFT which relates conformal QFTs to quantum gravity on anti de-Sitter spaces and this has limits that can be probed with QFT in curved space-time.

Quantization is a global problem, in which the global structure of space-time plays a crucial role. Thus the first half of the course will be devoted to improving our understanding of classical field theory in curved space-time and global features. Furthermore, Fermions play a basic role in physics, and require the use of spinors. We will therefore devote a couple of lectures to introducing spinors in curved space-time. These also have a number of independent applications such as the positive mass theorem and the geometry of congruences.

1.1 Conventions

Planck units $\hbar = c = G = k = 1 \rightsquigarrow$

- Mass $\sim 10^{-5} g \sim 10^{19}$ GeV.

- distance $\sim 10^{-33}$ cm

- time $\sim 10^{-44}$ sec

- temperature $\sim 10^{32}$ °K.

A nuclear mass $\sim 10^{-18}$, a Planck mass is almost visible. The cosmological constant is of the order of 3×10^{-122} in these units. For a body of mass M and size R, having

$$\frac{GM}{c^2R} > 1 \qquad \text{must use general relativity (GR)}$$
$$\frac{\hbar}{MRc} > 1 \qquad \text{must use quantum field theory (QFT)}\,.$$

This leads to a $M - R$-plane diagram of validity of theories.

Let (M, g_{ab}) be a space-time where for the most part, we will take M to be a 4-dimensional manifold, with local coordinates x^a, $a, b = 0, \dots, 3$ with metric g_{ab}. Indices are as usual raised and lowered by g_{ab} and its inverse g^{ab}.

We take Penrose conventions:
The metric has signature $(1, 3)$ The Ricci identity is

$$[\nabla_a, \nabla_b]V^d = R_{abd}{}^c V^d\,. \tag{1}$$

These conventions are best for spinors but a positive definite sphere has negative curvature (whereas a space-like sphere has positive curvature).[Another very common alternative is to have metric signature $(3, 1)$ and a minus sign in the above Ricci identity which conforms better with Riemmannian differential geometry, but less well with QFT and spinors.]

We then have for the Ricci curvature, scalar curvature and Einstein tensors respectively

$$R_{ab} = R_{acb}{}^c\,, \qquad R = R_a^a\,, \qquad G_{ab} = R_{ab} - \tfrac{1}{2} R g_{ab}\,. \tag{2}$$

The Einstein field equations are

$$G_{ab} + \lambda g_{ab} = -8\pi G T_{ab} \tag{3}$$

where λ is the cosmological constant, G Newton's constant and T_{ab} the stress-energy tensor.

As part of Penrose conventions, we have the abstract index notation. Indices are not understood to take on numeric values in general. They simply signify the type of tensor that the object is, having the same downstairs and upstairs indices as would be required if it were to be written out in a coordinate frame. To express a vector in some coordinate or frame basis, we underline the index to refer it to a basis. This avoids the ambiguity in the meaning of

$$\nabla_3 V^2$$

which could be $\partial_3 V^2$ or $\partial_3 V^2 + \Gamma^2_{3a} V^a$ because ∇_3 doesnt know whether to treat V^2 as a scalar or a component of a vector. So ∇ acting on an object with a numerical or concrete underlined index never uses the connection, whereas on an abstractly indexed quantity it does.

1.2 Some further geometry background: differential forms

Differential forms often simplify formulae both computationally and conceptually. A p-form $\alpha \in \Omega^p$ is a totally skew covariant tensor. We usually suppress the p skew downstairs indices by introducing formal objects dx^a so that

$$\alpha = \alpha_{a_1 a_2 \ldots a_p} dx^{a_1} \wedge \ldots dx^{a_p} = \alpha_{[a_1 a_2 \ldots a_p]} dx^{a_1} \wedge \ldots dx^{a_p} \in \Omega^p. \tag{4}$$

The $\wedge$ symbol signifies that the tensor is skew symmetrized, so that

$$dx^{a_1} \wedge \ldots \wedge dx^{a_p} = dx^{[a_1} \wedge \ldots \wedge dx^{a_p]} := \frac{1}{p!} \sum_{\sigma \in S_p} dx^{a_{\sigma(1)}} \wedge \ldots \wedge dx^{a_{\sigma(p)}}$$

In concrete indices these are just the infinitesimal coordinate variations $dx^{\underline{a}}$. There are two key operations with differential forms, the wedge product

$$\alpha \wedge \beta := \alpha_{[a_1 \ldots a_p} \beta_{a_{p+1} \ldots a_{p+q}]} dx^{a_1} \wedge \ldots \wedge dx^{a_{p+q}} \in \Omega^{p+q}\,, \tag{5}$$

where α is a p-form and β a q-form. This product is graded commutative

$$\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha\,. \tag{6}$$

We also have the exterior derivative defined by

$$d\alpha := dx^a \wedge \nabla_a \alpha\,. \tag{7}$$

Key features are:

Lemma 1.1 *The exterior derivative does not depend on the choice of torsion-free covariant derivative. We have $d^2\alpha = 0$ for all α asa consequence of the commutation of partial derivatives (or symmetry of a torsion-free connection).*

Thus it is metric independent and can be defined just using the coordinate derivative in any coordinate system. The fact that $d^2 = 0$ allows us to define cohomology groups

$$H^p(M) = \{\alpha \in \Omega^p | d\alpha = 0\}/\{\alpha = d\beta\}\,, \tag{8}$$

because the *exact* forms, those that can be expressed as $d\beta$, are a subset of the closed forms, those that satisfy $d\alpha = 0$. These encode the topology of M because $d\alpha = 0$ implies that locally there exists a β with $\alpha = d\beta$ (Poincaré lemma). As an example, consider $d\theta$ on the circle. Although clearly closed, $\theta \in \mathbb{R}/2\pi$ is not a single valued function on the circle, so it is not globally exact.

The exterior derivative satisfies the graded Leibnitz rule

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^p \alpha \wedge d\beta\,. \tag{9}$$

We also have the interior product with a vector V^a that takes a p-form α to a $p-1$-form

$$(V \lrcorner \alpha)_{a_2 a_3 \dots a_p} = p V^{a_p} \alpha_{a_1 \dots a_p}\,. \tag{10}$$

This also satisfies a graded leibnitz property,

$$V \lrcorner (\alpha \wedge \beta) = (V \lrcorner \alpha) \wedge \beta + (-1)^p \alpha \wedge (V \lrcorner \beta)\,. \tag{11}$$

It plays a role in the Cartan formula for the Lie derivative of a form

$$\mathcal{L}_V \alpha = V \lrcorner d\alpha + d(V \lrcorner \alpha)\,. \tag{12}$$

When we have a metric, we can define Hodge duality: in d dimensions a p-form α is dualized to a $d-p$ form ${}^*\alpha$ by

$$({}^*\alpha)_{a_{p+1} \dots a_d} := \frac{1}{p!} \varepsilon_{a_1 \dots a_d} \alpha^{a_1 \dots a_p} \tag{13}$$

where $\varepsilon_{a_1 \dots a_d} = \varepsilon_{[a_1 \dots a_d]}$ and $\varepsilon_{0\,1 \dots d-1} = \sqrt{-g}$ is the metric volume form.

A key application is to integration. Being a covariant tensor, a p-form naturally 'pulls back' under a map, and restricts to provide a p-form on a submanifold. On a p-dimensional submanifold, it can naturally be integrated subject to the choice of an orientation on the surface.

Definition 1.1 *A p-surface Σ^p is said to be orientable if it is possible to choose a non-vanishing p-form. Such a choice provides an* orientation *on Σ^p.*

The key point is that under a change of coordinates on the p-surface Σ^p, a p-form transforms with the determinant of the Jacobian of the coordinate transformation, whereas the change of variables formula for integration requires the modulus of the determinant which can introduce additional signs, and so we must restrict the coordinate transformations to those that preserve the sign of the chosen form making sure that the sign in question is positive.[1] The standard example of a non-orientable manifold is $\mathbb{RP}^{2n} = S^{2n}/\mathbb{Z}_2$ where the $\mathbb{Z}_2$ acts by the antipodal map which reverses the sign of the volume form.

The main theorem concerning integration on manifolds is Stoke's theorem:

Theorem 1 (Stokes) *Let Σ be a p-surface with boundary S with compatible orientations (i.e., the orientation on S is obtained from that on Σ by use of an outward pointing normal vector), and let α be a $p-1$-form on Σ, then*

$$\int_\Sigma d\alpha = \int_S \alpha \,. \tag{14}$$

Another application is the Cartan formulation of connections and curvature. We first choose an orthonormal frame of one-forms $e^{\underline{a}} := e_a^{\underline{a}} dx^a$ satisfying

$$g_{ab} = \eta_{\underline{ab}} e_a^{\underline{a}} e_b^{\underline{b}} \,, \tag{15}$$

where $\eta_{\underline{ab}} = \mathrm{diag}(1,-1,-1,-1)$ is the flat Lorentz metric. The $e_a^{\underline{a}}$ and its inverse $e_{\underline{a}}^{a}$ can be used to freely convert abastract indices into concrete indices and back again. The connection acting on this frame can be obtained from the Cartan structural equation

$$de^{\underline{a}} = \Gamma^{\underline{a}}{}_{\underline{b}} \wedge e^{\underline{b}} \tag{16}$$

where $\Gamma_{\underline{ab}} = \Gamma_{[\underline{ab}]} = dx^c \Gamma_{c\underline{ab}}$ are the connection 1-forms. These are as many equations as unknowns being 4 2-forms and are nondegenerate, so admit a unique solution for $\Gamma^{\underline{a}}{}_{\underline{b}}$. We can then define the full connection to be

$$\nabla_a e_c^{\underline{b}} = \Gamma_a{}^{\underline{b}}{}_{\underline{c}} e_c^{\underline{c}} \tag{17}$$

[1]The issue is seen in one dimension: under the transformation $y=-x$,

$$\int_a^b f(x)dx = \int_{-a}^{-b} -f(-y)dy = \int_{-b}^{-a} f(-y)dy \,,$$

so that there is no sign change if we are to integrate from the lower limit to the upper in each case.

so that for a general 1-form $A_a = e_a^{\underline{a}} A_{\underline{a}}$ we have

$$\nabla_a A_b = (\nabla_a A_{\underline{b}} - \Gamma_a{}^{\underline{c}}{}_{\underline{b}} A_{\underline{c}}) e_b^{\underline{b}}\,, \tag{18}$$

where according to the abstract index convention the first term is the ordinary derivative of the components of A_a and doesnt involve the connection. The skew symmetry of $\Gamma_a{}^{\underline{c}}{}_{\underline{b}}$ on its concrete indices then can be seen to be equivalent to the requirement that it preserves the metric $\nabla_a g_{bc} = 0$.

The connection 1-forms determine the curvature 2-form by

$$R_{\underline{a}}{}^{\underline{b}} := dx^c \wedge dx^d R_{cd\underline{a}}{}^{\underline{b}} = d\Gamma^{\underline{a}}{}_{\underline{b}} - \Gamma^{\underline{a}}{}_{\underline{c}} \wedge \Gamma^{\underline{c}}{}_{\underline{b}}\,. \tag{19}$$

which satisfy Bianchi identities

$$R^{\underline{a}}{}_{\underline{b}} \wedge e^{\underline{b}} = 0\,, \qquad dR^{\underline{a}}{}_{\underline{b}} + \Gamma^{\underline{a}}{}_{\underline{c}} \wedge R^{\underline{c}}{}_{\underline{b}} - \Gamma^{\underline{c}}{}_{\underline{b}} R^{\underline{a}}{}_{\underline{c}} = 0\,. \tag{20}$$

These essentially follow from $d^2 = 0$.

2 Classical fields in curved space-time

The main linear fields are Klein Gordon $\phi(x)$, Maxwell $A_a(x)$ and spinor fields (Dirac etc.). When coupling to a metric, we often adopt the minimal coupling prescription, that we take the flat space action, and replace coordinate derivatives by covariant derivatives sufficient to guarantee covariance. However, we could in principle include additional curvature terms if desired. For example, for scalar wave equation (Klein-Gordon) we can have

$$S[\phi] = \tfrac{1}{2} \int_M g^{ab} \partial_a \phi \partial_b \phi - (aR + m^2)\phi^2 \, \mathrm{d}\nu_g\,, \qquad \mathrm{d}\nu_g = \sqrt{-g} d^4 x\,. \tag{21}$$

Here m the mass and a is a number that can be zero, but when non-zero violates minimal coupling. This yields field equations

$$\left(\Box + m^2 + aR\right)\phi = 0\,. \tag{22}$$

However, the scalar curvature term has some utility because, when[2] and $m = 0$, this equation is conformally invariant under

$$(g_{ab}, \phi) \to \left(\Omega^2 g_{ab}, \frac{\phi}{\Omega}\right)\,, \tag{23}$$

[2]in d dimensions, when $a = (d-2)/4(d-1)$, although with a different scaling weight.

for any $\Omega(x) \neq 0$.

We remark on the differential form version of the kinetic term

$$\int d\phi \wedge {}^*d\phi \tag{24}$$

which leads to the coordinate formula for the wave operator

$$\Box\phi = {}^*(d^*d\phi) = \frac{1}{\sqrt{-g}}\partial_{\underline{a}}(\sqrt{-g}g^{\underline{ab}}\partial_{\underline{b}}\phi) = \nabla_a\nabla^a\phi\,. \tag{25}$$

We determine the energy momentum tensor by

$$T^{ab} = -\frac{\delta S}{\delta g_{ab}} \tag{26}$$

so as to give the source term for the Einstein equations. This yields for $a = 0$

$$T_{ab} = \partial_a\phi\partial_b\phi - \frac{1}{2}g_{ab}((\partial\phi)^2 - m^2\phi^2)\,.$$

For an observer with 4-velocity U^a, the field has 4-momentum density $T_{ab}U^b$.

Differential forms come into their own in Maxwell theory. These are equations on a 1-form potential $A = A_a dx^a \in \Omega^1$ defined up to the gauge freedom $A_a \to A_a + \partial_a g(x)$, or $A \to A + dg$, for arbitrary $g(x)$. The field is

$$F = F_{ab}dx^a \wedge dx^b = dA \in \Omega^2\,, \qquad F_{ab} = \nabla_{[a}A_{b]}\,, \tag{27}$$

and the action coupled to gravity is

$$S[A_a] = \frac{1}{4}\int_M F \wedge {}^*F = \frac{1}{2}\int_M F_{ab}F^{ab}d\nu_g\,, \tag{28}$$

with Bianchi identities $dF = 0$, or $\nabla_{[a}F_{bc]} = 0$ and field equations

$$d^*F = 0\,, \qquad \text{or} \qquad \nabla^a F_{ab} = 0\,. \tag{29}$$

In order to obtain a deterministic equation, it is normal to impose Lorenz gauge $\nabla^a A_a = 0$ upon which these equations reduce to the wave equation $\Box A_a = 0$ although there is nevertheless still residual gauge freedom under $A \to A + dg$ with $\Box g = 0$. These equations are conformally invariant (see problem sheet).

The stress-energy tensor in this case is

$$T_{ab} = F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F_{cd}F^{cd} \,. \tag{30}$$

Conformal invariance here is manifested in the fact that $T^a_a = 0$. This is a general property of conformally invariant field theories, and is a consequence of the invariance of the action under $\delta g_{ab} = \omega g_{ab}$.

In both cases we have the positivity of energy manifested in the *dominant energy* condition that for any non-zero timelike or null vector t^a,

$$T_{ab}t^a t^b \geq 0 \tag{31}$$

with equality in the timelike case if only if $F_{ab} = 0$ or $\nabla_a \phi = 0$.

2.1 Spinors and space-time

To discuss the Dirac and Rarita-Schwinger equations in curved space-time we need to introduce spinors. In flat space we introduce 2-component spinors via the identification of $\mathbb{R}^4$ with Hermitian 2×2 matrices:

$$dx^{AA'} := \sigma_a^{AA'} dx^a := \frac{1}{\sqrt{2}} \begin{pmatrix} dt + dz & dx + idy \\ dx - idy & dt - dz \end{pmatrix} , \qquad A = 0, 1, A' = 0', 1' \,. \tag{32}$$

The matrices $\sigma_a^{AA'}$ are sometimes known as Van de Waerden symbols. The determinant is a multiple of the metric. This can be expressed by introducing

$$\varepsilon_{AB} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \qquad \varepsilon_{A'B'} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \tag{33}$$

so that

$$ds^2 = \eta_{ab} dx^a dx^b = \varepsilon_{AB}\varepsilon_{A'B'} dx^{AA'} dx^{BB'} \,. \tag{34}$$

We use the ε_{AB} and its inverse ε^{AB} to raise and lower indices via

$$\psi^A \varepsilon_{AB} = \psi_B \,, \qquad \psi^A = \varepsilon^{AB}\psi_B \,, \tag{35}$$

and similarly for the primed version; beware signs, particularly when differentiating with respect to spinors.

Let $\mathbb{S}$ denote the two-dimensional complex vector space of *spinors* ψ^A and $\mathbb{S}'$ primed spinors $\psi^{A'}$. Although the above has been written out in a concrete basis, it can be undertood to express the abstract isomorphism

$$T = \mathbb{S} \otimes \mathbb{S}' \,, \tag{36}$$

where T here is the tangent space. We will often use this to replace vector indices by pairs of spinor indices all thought of as abstract indices[3] so we can write for abstract indices only

$$V^a = V^{AA'}\,. \tag{37}$$

The above establishes the Lorentz invariant identification of $(\mathbb{R}^4, \eta_{ab})$ with $(\mathbb{S}\otimes\mathbb{S}', \varepsilon_{AB}, \varepsilon_{A'B'})$ underpinned by the spinor isomorphism between the space and time orientation preserving Lorentz group $SO_+(1,3)$ and $SL(2,\mathbb{C})/\mathbb{Z}_2$ given by

$$L_a^b \sigma_b^{AA'} = L_B^A \bar{L}_{B'}^{A'} \sigma_a^{BB'}\,, \quad L_b^a \in SO_+(1,3)\,, \quad L_B^A \in SL(2,\mathbb{C})\,. \tag{38}$$

Since primed spinors transform with the complex conjugate $SL(2,\mathbb{C})$ there is a complex conjugation map

$$\overline{\mathbb{S}} = \mathbb{S}'\,, \qquad \psi^A \to \bar{\psi}^{A'}\,. \tag{39}$$

For infinitesimal Lorentz transformations $l_{ab} = l_{[ab]}$, this is given in spinors by

$$l^{ab}\sigma_a^{AA'}\sigma_b^{BB'} = l^{AA'BB'} = \varepsilon^{A'B'} l^{AB} + \varepsilon^{AB}\bar{l}^{A'B'} \tag{40}$$

where

$$l^{AB} = l^{(AB)} = \tfrac{1}{2} l^{AA'BB'}\varepsilon_{A'B'} \tag{41}$$

so that on the Lie algebra level $so(1,3) = sl(2,\mathbb{C}) \oplus \overline{sl(2,\mathbb{C})}$.

To prove this note first that $\psi^{AB} - \psi^{BA} = \varepsilon^{AB}\psi_C{}^C$ as skew matrices in 2d are necessarily multiples of ε^{AB}. We can use the skew symmetry of l_{ab} to write

$$l^{AA'BB'} = \tfrac{1}{2} l^{AA'BB'} - \tfrac{1}{2} l^{BA'AB'} + \tfrac{1}{2} l^{AA'BB'} - \tfrac{1}{2} l^{AB'BA'}. \tag{42}$$

where the two terms with minus signs are equal and opposite by skew symmetry of l^{ab}. The first pair of terms therefore reduces to $\varepsilon^{AB}\bar{l}^{A'B'}$ and the second its conjugate.

These are reduced (chiral) spinors. They are related to Dirac spinors by $\psi^\alpha = (\psi^A, \phi^{A'})$. The Clifford matrices are represented in terms of Van de Waerden symbols by

$$\gamma^\alpha_{c\beta} = \sqrt{2}\begin{pmatrix} 0 & \sigma_{cB'}{}^A \\ \sigma_{cB}{}^{A'} & 0 \end{pmatrix}\,, \qquad \gamma_a\gamma_b + \gamma_b\gamma_a = -2I\eta_{ab} \tag{43}$$

[3]this doesnt work well in dimensions greater than 6.

suppressing the the Dirac spinor indices. The Dirac equation $\gamma^a\partial_a\psi = m\psi$ in this notation becomes

$$\partial_{AA'}\psi^A = m\phi_{A'}\,, \qquad \partial_{AA'}\phi^{A'} = m\psi_A\,, \tag{44}$$

where we have introduced the notation $\partial_{AA'} = \sigma^a_{AA'}\partial_a$.

This can be extended to curved space by introducing an orthonormal tetrad $e^{\underline{a}}_a := (e^0_a, e^1_a, e^2_a, e^3_a)$ such that

$$g_{ab} = \eta_{\underline{ab}} e^{\underline{a}}_a e^{\underline{b}}_b\,. \tag{45}$$

We can then use $\sigma_{\underline{a}}^{\underline{AA'}}$ to introduce spinors with respect to the orthonormal frame.

To extend the Dirac equation to curved space, we must introduce covariant differentiation for spinors. In an orthonormal frame we introduce the Ricci rotation coefficiencts via

$$\nabla_b e^{\underline{a}}_c = \Gamma_b{}^{\underline{a}}{}_{\underline{c}} e^{\underline{c}}_c\,, \tag{46}$$

where ∇_b is the covariant derivative, and the abstract index notation is now being used to indicate that the derivative uses the space-time connection on abstract but not concrete indices. Since $\nabla_a g_{bc} = 0$ and $\eta_{\underline{ab}}$ are constant,

$$\Gamma_{b\underline{ac}} = \Gamma_{b[\underline{ac}]} \tag{47}$$

and so converting to spinors using the Van de Waerden symbols we can define the spin connection by

$$\Gamma_{b\underline{AB}} = \tfrac{1}{2}\Gamma_{b\underline{AA'}\underline{B}}{}^{\underline{A'}}\,, \qquad \Gamma_{b\underline{A'}\underline{B'}} = \tfrac{1}{2}\Gamma_{b\underline{A'}\underline{AB'}}{}^{\underline{A}}\,. \tag{48}$$

These define the covariant derivative of the spin frame $\varepsilon_A{}^{\underline{A}}$ that corresponds to our choice of orthonormal frame by

$$\nabla_a \varepsilon_A{}^{\underline{A}} = \Gamma_{a\underline{B}}{}^{\underline{A}} \varepsilon_A{}^{\underline{B}} \tag{49}$$

and this together with the complex conjugate determines the covariant derivatives on all spinors by the relations

$$\varepsilon_B{}^{\underline{B}} \nabla_a \alpha^B = \nabla_a \alpha^{\underline{B}} - \Gamma_{a\underline{C}}{}^{\underline{B}} \alpha^{\underline{C}}\,. \tag{50}$$

where the first term on the right, according to the abstract index convention denotes the ordinary derivative of the components of $\alpha^{\underline{B}}$ whereas on the left, $\nabla_a\alpha^B$ is necessarily a covariant derivative.

Once we are happy using fully abstract indices, we can incorporate the isomorphism $TM = \mathbb{S} \otimes \mathbb{S}'$ given by the abstract $\sigma_a^{AA'}$ into equations writing for example

$$\nabla_a \alpha^B = \nabla_{AA'} \alpha^B \,. \tag{51}$$

The curvature on spinors is given by spinorial Ricci identities

$$[\nabla_{AA'}, \nabla_{BB'}]\alpha^C = \left(\varepsilon_{A'B'}\left(\Psi_{ABD}{}^C - \frac{R}{12}\varepsilon_{D(A}\varepsilon_{B)}{}^C\right) + \varepsilon_{AB}\Phi_{A'B'D}{}^C\right)\alpha^D \tag{52}$$

Here R is the scalar curvature,

$$\Phi_{ABA'B'} = \Phi_{(AB)(A'B')} \,, \qquad \Phi_{ab} = -\frac{1}{2}\left(R_{ab} - \frac{R}{4}g_{ab}\right) \tag{53}$$

the trace-free Ricci curvature, and $\Psi_{ABCD} = \Psi_{(ABCD)}$ is the spinorial version of the Weyl curvature

$$C_{abcd} = \varepsilon_{A'B'}\varepsilon_{C'D'}\Psi_{ABCD} + \varepsilon_{AB}\varepsilon_{CD}\bar{\Psi}_{A'B'C'D'} \,. \tag{54}$$

It is also called the conformal curvature because $C_{abc}{}^d$ is invariant under $g_{ab} \to \Omega^2 g_{ab}$. It can be written in terms of the regular curvature as

$$C_{ab}{}^{cd} = R_{ab}{}^{cd} - 4P_{[a}^{[c}\delta_{b]}^{d]} \tag{55}$$

where P_{ab} is the Schouten tensor

$$P_{ab} = -\frac{1}{2}R_{ab} + \frac{1}{12}Rg_{ab} \,, \tag{56}$$

which we will see later because of its good conformal variations properties.

In this notation we have the Bianchi identities

$$\nabla_{A'}^D \Psi_{ABCD} = \nabla_{(A}^{B'}\Phi_{BC)A'B'} \qquad \nabla^a\Phi_{ab} + \nabla_b R/8 = 0 \tag{57}$$

The Massless field equations: We can now write down massless field equations for arbitrary half-integral helicity s on space-time. These are equations on a symmetric spinor field $\phi_{A_1A_2\ldots A_{2s}} = \phi_{(A_1A_2\ldots A_{2s})}(x)$

$$\nabla_{A'}^{A_1}\phi_{A_1\ldots A_{2s}} = 0 \,. \tag{58}$$

For $s < 0$ we have the complex conjugate equation on primed spinors (and at $s = 0$ the scalar wave equation). We have key examples:

1. $s=1/2$, the Weyl neutrino equation (chiral massless Dirac).
2. $s=1$ we obtain the spinor form of the Maxwell Field equations

$$F_{ab}=\varepsilon_{A'B'}\phi_{AB}+\varepsilon_{AB}\bar{\phi}_{A'B'}\,. \tag{59}$$

   The Maxwell equations $\nabla^a F_{ab}=0$, and $\nabla_{[a}F_{bc]}=0$ become

$$\nabla^{AA'}\phi_{AB}=0. \tag{60}$$

   To see this it is helpful to note that under *Hodge duality* we have

$$\tfrac{1}{2}\varepsilon_{ab}{}^{cd}F_{cd}=i\varepsilon_{A'B'}\phi_{AB}-i\varepsilon_{AB}\bar{\phi}_{A'B'}\,. \tag{61}$$

   which follows from the expression

$$\varepsilon_{abcd}=i\varepsilon_{AC}\varepsilon_{BD}\varepsilon_{A'D'}\varepsilon_{B'C'}-i\varepsilon_{AD}\varepsilon_{BC}\varepsilon_{A'C'}\varepsilon_{B'D'}\,. \tag{62}$$

   Thus ϕ_{AB} defines a self-dual two form ($+i$ eigenvalue under Hodge duality) and $\bar{\phi}_{A'B'}$ anti-self-dual.

3. For $s=2$ we obtain the vacuum Bianchi identity on the Weyl Spinor

$$\nabla^{AA'}\Psi_{ABCD}=0\,, \tag{63}$$

   thus describing gravity.

There are two key results for the general massless field equations

Proposition 2.1 *The massless field equations are conformally invariant under $g_{ab}\to\Omega^2 g_{ab}$ with $\phi_{A_1\dots A_{2s}}\to\Omega^{-1}\phi_{A_1\dots A_{2s}}$.*

Proof: by direct calculation using the conformal variation formulae under $g_{ab}\to\hat{g}_{ab}=\Omega^2 g_{ab}$ that give $\nabla_a\to\hat{\nabla}_a$ such that

$$\hat{\nabla}_{AA'}\xi^{B'\dots}_{B\dots}=\nabla_{AA'}\xi^{B'\dots}_{B\dots}-\Upsilon_{A'B}\xi^{B'\dots}_{A\dots}-\dots+\varepsilon_{A'}{}^{B'}\Upsilon_{AC'}\xi^{C'\dots}_{B\dots}+\dots\,, \tag{64}$$

where $\Upsilon_a=\nabla_a\log\Omega$ and the $\dots$ can include terms with primed exchanged by unprimed indices in the obvious way, with one term for each index on ξ. These formulae can be obtained for example from the Cartan structure equations. If all the indices are downstairs and in addition $\hat{\phi}_{B_1\dots B_n}=\phi_{B_1\dots B_n}/\Omega$ we have

$$\Omega\hat{\nabla}_{AA'}\hat{\phi}_{B_1\dots B_n}=\nabla_{AA'}\phi_{B_1\dots}-\Upsilon_{AA'}\phi_{B_1\dots}-\Upsilon_{A'B_1}\phi_{AB_2\dots}-\dots\,, \tag{65}$$

and so the resulting expression is symmetric in its unprimed indices and so will vanish on contraction with the skew ε^{AB_1}. □

Note that the law satisfied by the Weyl spinor is that under $g_{ab} \to \Omega^2 g_{ab}$, $\Psi_{ABCD} \to \Psi_{ABCD}$ and not the spin-2 variation given above, so Einstein's field equations are not conformally invariant as could be expected. Nevertheless, the vacuum Bianchi identity allows us to rescale the Weyl spinor into a solution to the conformally invariant spin-2 equation.

Proposition 2.2 *The massless field equations* (58) *with* $s > 1$ *are overdetermined and inconsistent.*

Proof: A symmetric spinor $\phi_{A_1 \dots A_{2s}}$ has $2s+1$ components whereas there are $2 \times 2s$ equations, i.e., a surfeit of $2s-1$ equations. These only lead to problems in curved space where taking a further derivative of the equation and using the Ricci-identities (52) we obtain

$$0 = \nabla^{A_1}_{A'} \nabla^{A_2 A'} \phi_{A_1 \dots A_{2s}} = \Psi^{B_1 B_2 B_3}{}_{(A_3} \phi_{A_4 \dots A_{2s}) B_1 B_2 B_3} \,. \tag{66}$$

This is vacuous in spin one giving a Bianchi identity amongst between the field equations (corresponding to charge conservation) and it then gives Bianchi identities showing the equations are consistent, but with higher spin in curved space it implies new relations on the fields and the equations rapidly become inconsistent. □

The Weyl tensor itself escapes via an algebraic identity that gives the automatic vanishing of the RHS when $\Psi_{ABCD} = \phi_{ABCD}$. Otherwise, spin-2 fields are inconsistent on curved space.

Spin 3/2 fields are a key ingredient of supergravity theories. They escape the Buchdahl conditions in a Ricci-flat background via the Rarita-Schwinger equation, a potential modulo gauge version appropriate for gauging the supersymmetry. This is best understood as an analogue of a Maxwell potential

$$\rho_A = dx^b \rho_{bA} \,, \qquad \text{modulo gauge freedom} \qquad \delta\rho_A = d\xi_A := dx^b \nabla_b \xi_A \,. \tag{67}$$

The action is

$$S = \int_M i \bar{\rho}_{A'} \wedge dx^{AA'} \wedge d\rho_A \,, \tag{68}$$

which gives the field equations

$$dx^{AA'} \wedge d\rho_A := dx^d \wedge dx^c \wedge dx^b \sigma^{AA'}_{[d} \nabla_c \rho_{b]A} = 0 \,. \tag{69}$$

Here $\sigma_b^{AA'}$ are the abstract Van der Waerden symbols. In flat space, this relates to the spin 3/2 field above because a consequence of this equation is that

$$d\rho_A = \phi_{ABC}\varepsilon_{B'C'}dx^{BB'} \wedge dx^{CC'} \,, \tag{70}$$

where ϕ_{ABC} is a spin 3/2 massless field in the sense above. However, in curved space, a pure gauge field gives¡ using the Ricci identity (52) with vanishing Ricci tensor in differential form version gives

$$d^2\xi_A = -dx^{BB'} \wedge dx^{CC'}\varepsilon_{B'C'}\Psi_{ABC}{}^D\xi_D \,,$$

so that ϕ_{ABC} is not a gauge invariant quantity, changing by $\Psi_{ABC}{}^D\xi_D$.

It is nontrivial that the field equation is compatible with the gauge freedom, but we have the identity

$$d\left(dx^{AA'} \wedge d\sigma_A\right) = dx^{AA'} \wedge d^2\sigma_A = iG_b^{AA'}{}^*dx^b \wedge \sigma_A \,. \tag{71}$$

Here ${}^*dx^a = \varepsilon^a_{bcd}dx^b \wedge dx^c \wedge dx^d$, and $d^2 \neq 0$ because it is acting on an abstractly indexed quantity and hence requires a commutator that gives rise to curvature. Since G_{ab} is the Einstein tensor, in vacuum we can take ρ_A to be a 0-form, hence proving the consistency of the gauge freedom with the field equations, or as a 1-form, providing a Bianchi identity that gives the consistency of the overdetermined field equations amongst themselves. These identities play a key role in Witten's positive mass theorem.

2.2 Null congruences

A null congruence is a foliation of a region of space-time by null geodesics. It can be defined by a null vector field l^a whose integral curves are the null geodesics through each point. If it is tangent to a congruence of affinely parametrised null geodesics, then

$$\nabla_l l^b := l^a\nabla_a l^b = 0 \tag{72}$$

Spinors are particularly natural for describing null congruences because a 4-vector l^a is null iff it can be expressed as $l^a := o^A\bar{o}^{A'}$. This follows because the vanishing determinant of $l^{AA'}$ implies that it has rank 1 and conversely. It is always possible to choose the phase of o^A so that o^A is parallel also

$$o^A\bar{o}^{A'}\nabla_{AA'}o^B = 0 \,. \tag{73}$$

It is a standard fact that for spinors α^A, β^A, $\alpha^A\beta_A = 0$ iff they are proportional

$$\alpha^A = f\beta^A \tag{74}$$

for some f (i.e., they are proportional) as spin space is two-dimensional and the inner product skew. We can deduce that there is a pair of complex scalars ρ, σ such that

$$o_B\bar{o}^{A'}\nabla_{AA'}o^B = -\rho o_A\,, \quad o_B o^A\nabla_{AA'}o^B = -\sigma\bar{o}_{A'}\,. \tag{75}$$

These have the following geometric interpretation: parametrize the two-plane orthogonal and transverse to l^a by $\zeta \in \mathbb{C}$ by

$$X^a = \zeta\bar{m}^a + \bar{\zeta}m^a\,, \qquad m^a := o^A\bar{\iota}^{A'} \tag{76}$$

for some choice of ι^A with $o_A\iota^A = 1$, and m^a is a complex null vector defined modulo l^a. We can choose ι^A so that $\nabla_l\iota^A = 0$, and then we will also have $\nabla_l m^a = 0$. If X^a connects nearby geodesics of the congruence, then it is Lie derived along l^a, i.e.,

$$[l, X]^a = \nabla_l X^a - \nabla_X l^a = 0\,. \tag{77}$$

This gives

$$\nabla_l\zeta = -\rho\zeta - \sigma\bar{\zeta}\,. \tag{78}$$

This can be interpreted as follows:

1. The imaginary part of ρ is the twist and generates rotations of the ζ plane. It vanishes iff the congruence is hypersurface forming, $l_{[a}\nabla_b l_{c]} = 0$ which implies that there is a rescaling of o^A so that $o_A o_{A'} = \nabla_{AA'}u$ for some function u.

2. The real part of ρ gives the *expansion*, $\nabla_a l^a = -2\rho$ and the area element of the orthogonal transverse plane evolves by

   $$A = -im_a dx^a \wedge \bar{m}_b dx^b,$$

   satisfies

   $$\mathcal{L}_l A = -2\rho A \tag{79}$$

3. The complex scalar σ is the shear in the sense that a circle in the ζ plane evolves into an ellipse.

4. Equation (77) implies the geodesic deviation equation
$$\nabla_l \nabla_l X^a = l^b l^c X^d R_{bdc}{}^a \tag{80}$$
and this combines with (78) to give the *Sachs equations*
$$\begin{aligned} \nabla_l \rho &= \rho\bar{\rho} + \sigma\bar{\sigma} + \Phi_{00} &(81)\\ \nabla_l \sigma &= (\rho + \bar{\rho})\sigma + \sigma\bar{\sigma} + \Psi_0 &(82)\end{aligned}$$
Here $\Psi_0 = \Psi_{ABCD} o^A \dots o^D$, $\Phi_{00} = \Phi_{ab} l^a l^b = -\frac{1}{2} R_{ab} l^a l^b$ and is positive when the dominant energy condition is satisfied. An important consequence for horizons and singularity theorems is that the whole RHS of (81) is manifestly positive definite.

5. If a null hypersurface has vanishing shear, then it has the intrinsic geometry of a light cone or null hyperplane in Minkowski space up to scaling (i.e. the metric restricts to a multiple of $d\zeta d\bar{\zeta}$ on $\mathbb{R}^3$ or $S^2 \times \mathbb{R}$ where $l^a \partial_a = \partial_v$ for a third coordinate v).

3 Causal structure and global hyperbolicity

The wave equation $\Box\phi = 0$ is hyperbolic and, in the massless case, propagates data along null geodesics, see for example the flat space solutions $f(k_a x^a) = f(t - z)$ where $k_a = (1, 0, 0, 1)$ is a null vector and f an arbitrary wave profile (more generally with some back-reaction, information propagates along causal curves).

Unless otherwise stated, we will take space-time to be both space-time and time orientable and oriented, i.e., we can consistently pick a future directed component of the lightcone at each point, and a non-vanishing four-form.

Time orientability isn't quite enough to rule out almost timelike curves which could be quite bad so we assume

Definition 3.1 *A spacetime (M, g) is* strongly causal *if for all $p \in M$ there exists an open neighbourhood U of p such that no causal (i.e., timelike or null) curve intersects V more than once.*

We expect to be able to solve an initial value problem (IVP) in which we pose initial data $(\phi, \dot{\phi})$ on some space-like[4] 3-surface $\Sigma \subset M$ and let the equation evolve ϕ off the surface. More properly, we will require

[4]Characteristic initial value problems can also be considered on null hypersurfaces, although the nature of the initial data changes there.

Definition 3.2 *a hypersurface* Σ *is* achronal *if no pair of points in* Σ *can be connected by a timelike curve.*

We will say that the initial value problem for solutions on some region U with the given data on Σ is *well posed* if there exists a unique solution on U with given data[5] on Σ.

The fact that solutions propagate along null or timelike curves (i.e., causal curves) suggests that the data on Σ can only influence the region

$$J^+(\Sigma) = \{p \in M | \exists \text{ future directed causal curve from } \Sigma \text{ to } p\} . \qquad (83)$$

This is the future of the set Σ and can be defined for any type of set. $J^+(\Sigma)$ is also said to be the domain of influence of Σ. We can similarly define the past of Σ,

$$J^-(\Sigma) = \{p \in M | \exists \text{ future directed causal curve from } p \text{ to } \Sigma\} . \qquad (84)$$

and one uses $I^{\pm}$ replacing causal by strictly timelike. These sets are the interiors of the $J^{\pm}$.

Definition 3.3 *The future domain of dependence* $D^+(\Sigma)$ *of* Σ *is*

$$D^+(\Sigma) = \{p \in M | \textit{every past inextendible causal curve from } p \textit{ intersects } \Sigma\} . \qquad (85)$$

Replacing past by future, we similarly define $D^-(\Sigma)$ *and the full domain of dependence by* $D(\Sigma) = D^+(\Sigma) \cup D^-(\Sigma)$.

This is the region on which the initial value problem for wave equations can be proved to be well-posed by PDE techniques. If p is a point lying on a causal curve that cannot be extended in the past through Σ, then one can envisage waves coming in along that curve that are not determined by data on Σ and so would violate the uniqueness assumption.

Definition 3.4 *A spacelike hypersurface* Σ *is a Cauchy surface for* M *if* $D(\Sigma) = M$. *A space-time is said to be globally hyperbolic if it admits a Cauchy surface.*

We have

[5]The solution is also usally required to depend continuously on the data, although this is straightforward for linear equations.

Theorem 2 (Geroch 1970) *If (M, g_{ab}) is globally hyperbolic, with Cauchy hypersurface Σ then M is diffeomorphic to $\Sigma \times \mathbb{R}$ with the second factor determined by a smooth time coordinate t such that each Σ_t is a Cauchy surface.*

We quote that the the IVP for linear wave equations of the form $\Box\phi + V(x)\phi = f(x)$ are well posed with data given by $(\phi, \dot{\phi})$ in Sobolev spaces and other function spaces on a Cauchy hypersurface in globally hyperbolic M. The proofs usually proceed by energy estimates.

4 Conformal infinity and Penrose diagrams

To obtain a good grip on the global structure, one needs to understand asymptotics. A neat way to do that is via *conformal compactification*, which involves adding a conformal boundary to space-time the corresponds to infinity in the physical space-time.

A key feature of the diagrams that we will draw is that they represent the causal structure directly by drawing light rays at 45 degrees. They will give an intuition for the asymptotics by bringing infinity into the finite part of the diagram so that we can see which light rays go where. Such diagrams are known as *Penrose diagrams* (or Penrose-Carter diagrams in Cambridge).

4.1 The homogeneous cases

A first example is $\mathbb{C} \to \mathbb{CP}^1 = S^2$ by stereographic projection and this extends in Euclidean signature to $\mathbb{R}^n \to S^n$. Here coordinates near infinity are mapped to those near the origin via the inversion

$$x^a \to \tilde{x}^a = \frac{x^a}{x^2}\,, \qquad x^2 := x^a x_a\,, \tag{86}$$

under which

$$ds^2 = dx^a dx_a = \frac{d\tilde{x}^a d\tilde{x}_a}{(\tilde{x}^2)^2}\,. \tag{87}$$

Such a transformation that preserves the metric up to a rescaling $g \to \Omega^2 g$ is said to be a conformal motion. Here the rescaling $\Omega = \tilde{x}^2$ returns the RHS to manifest flatness.

The same formulae hold in Lorentz signature, but now the light cone $x^2 = 0$ of the origin $x^a = 0$ is sent to infinity, being interchanged with the light cone $\tilde{x}^2 = 0$ of the point i at infinity given by $\tilde{x}^a = 0$, not just the points. Notice that the scale factor $\Omega = \tilde{x}^2$ vanishes on this light cone at infinity to first order.

To be more systematic, we introduce the full conformal group of (conformally) flat space-time but this can only act on a compactification as it interchanges finite with infinite points. We will denote points at infinity by i and hypersurfaces at infinity by $\mathscr{I}$, pronounced *scri* for script I.

For a flat metric of signature (p, q) the full conformal group is $SO(p + 1, q + 1)/\mathbb{Z}_2$, and so in four dimensions with Lorentz signature we have the 15 parameter group $SO(2, 4)$. This acts on $\mathbb{R}^6$ with coordinates

$$X^I = (s, w, x^a) = (t, x, y, z, s, w)\,, \qquad a =, 0, \ldots, 3\,, \quad I = 0, \ldots, 5,$$

by orthogonal transformations preserving the quadratic form

$$X^2 := \eta_{IJ} X^I X^J = s^2 - w^2 + x^a x_a\,. \tag{88}$$

Define first the projective space

$$\mathbb{RP}^5 = \mathbb{R}^6 / \{X^I \sim \lambda X^I, \lambda \in \mathbb{R} - \{0\}\}\,. \tag{89}$$

Then we can define conformally compactified Minkowski space to be

$$\mathbb{M} = \{[X^I] \in \mathbb{RP}^5 | X^2 = 0\} \subset \mathbb{RP}^5\,. \tag{90}$$

Lemma 4.1 $\mathbb{M} = S^1 \times S^3/\mathbb{Z}_2$.

This follows by rewriting $X^2 = 0$ and rescaling so that

$$s^2 + t^2 = w^2 + x^2 + y^2 + z^2 = 1\,.$$

Thus (s, t) lie on S^1 and (w, x, y, z) on S^3 with the Cartesian product metric, although note that $X^I \sim -X^I$, hence the $\mathbb{Z}_2$.□

The Einstein cylinder: We can take the universal cover by unwrapping the 'time' S^1 by setting $(s, t) = (\cos\tau, \sin\tau)$. This then gives the *Einstein cylinder* metric

$$ds^2_{EC} = \mathrm{d}\tau^2 - \mathrm{d}s^2_{S^3} \tag{91}$$

where the unit round sphere 3-metric can be given in spherical polars by

$$ds^2_{S^3} = \mathrm{d}\psi^2 + \sin^2\psi(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\,, \qquad (\psi, \theta, \phi) \in [0, \pi] \times [0, \pi] \times [0, 2\pi] \tag{92}$$

where $(w, x, y, z) = (\cos\psi, \sin\psi \sin\theta \cos\phi, \sin\psi \sin\theta \sin\phi, \sin\psi \cos\theta)$, although there will be coordinate singularities at $\theta, \psi = 0, \pi$. We either draw this as a cylinder, or in the (τ, ψ)-strip $\mathbb{R} \times [0, \pi]$.

To obtain the key maximally symmetric examples, we choose a non zero constant vector $K^I \in \mathbb{R}^6$ and define

$$ds^2_{K^2} = \left. \frac{\eta_{IJ} dX^I dX^J}{(K \cdot X)^2} \right|_{X^2=0}, \tag{93}$$

where $K \cdot X := K_I X^I$, $X^2 = X_I X^I$ etc.; under $SO(2, 4)$, K^I is distinguished only by its norm K^2 so there are only the 3 cases $K^2 = -1, 0, 1$. With maximal symmetry there is only the scalar curvature (the Weyl tensor and trace-free Ricci tensor must vanish) and its sign is that of K^2. By dividing by a quadratic function, the metric is invariant under constant rescalings of X^I. However, on $X^2 = 0$ the form $X_I dX^I = dX^2/2$ vanishes so it is easy to see that under $X^I \to f(X)X^I$, ds^2_I is invariant for any $f(X)$. Thus we can scale X so that $K \cdot X = 1$.

From the inversion example we see that the set where $K \cdot X = 0$ will correspond to points at infinity. These sets will be denoted $\mathscr{I}$, or $\mathscr{I}^+$ and $\mathscr{I}^-$ if respectively to the future or past of the finite part of space-time where $K \cdot X$ can be scaled to be 1. The isometry group of $ds^2_{K^2}$ is the subgroup of $SO(2, 4)$ that preserves K^I.

There are three cases:

$K^2 = 0$ *Flat space*. We can take $K = (0, 0, 0, 0, 1, -1)$, so that $K \cdot X = s + w = 1$. It is then immediate that $X^2 = 0$ gives $s - w = -x^a x_a$ and

$$ds^2_0 = dx^a dx_a \tag{94}$$

i.e., flat space as desired. Thus (93) defines a conformally flat metric.

Had we chosen to rescale so that $s + w = 1$ instead (but with the same K^I), we would have obtain the inverted metric given by the right hand side of (87).

To rewrite this in terms of Einstein cylinder coordinates we must divide (91) by $s + w = \cos\tau + \cos\psi = 2\cos(\frac{\psi+\tau}{2})\cos(\frac{\tau-\psi}{2})$

$$\mathrm{d}s^2_0 = \frac{\mathrm{d}s^2_{EC}}{4\cos^2(\frac{\psi+\tau}{2})\cos^2(\frac{\tau-\psi}{2})} \tag{95}$$

Future infinity $\mathscr{I}^+$ is null defined by $\tau + \psi = \pi$; it is the past lightcone of i^+ with $(\tau, \psi) = (\pi, 0)$ or future lightcone of i^0 with $(\tau, \psi) = (0, \pi)$. Past infinity $\mathscr{I}^-$ is $\tau - \psi = -\pi$ and is the past light cone of i^0 and future lightcone of i^- given by $(\tau, \psi) = (-\pi, 0)$. In $\mathbb{M}$, the three points i^0, i^+ and i^- are identified, and $\mathscr{I}^+$ is identified with $\mathscr{I}^-$.

$K^2 = 1$ *De Sitter space*; Einstein vacuum with cosmological constant $+1$ and isometry group $SO(1,4)$.

Put $K = (0,0,0,0,1,0)$ so $s = 1$ and $X^2 = 0$ gives the hyperboloid

$$1 + t^2 = w^2 + x^2 + y^2 + z^2 \tag{96}$$

This clearly has topology $\mathbb{R} \times S^3$, with 3-spheres of radius $r = \sqrt{1+t^2}$ at time t. Introducing $(t, r) = (\tan\tau, \sec\tau)$ we can rewrite the metric as

$$ds_1^2 = \frac{1}{\cos^2\tau}\left(d\tau^2 - ds_{S^3}^2\right) \tag{97}$$

This is therefore the region $\tau \in [-\pi/2, \pi/2]$ in the Einstein cylinder with future/past infinities $\mathscr{I}^\pm$ both of topology S^3 given by $\tau = \pm\pi/2$.

Alternative coordinates $(t, r) = (\sinh T, \cosh T)$ yield

$$ds_1^2 = dT^2 - \cosh^2 T ds_{S^3}^2 \tag{98}$$

emphasizing the hyperbola shape with exponential expansion as appropriate for inflationary cosmology. Here T is proper time for observers fixed in S^3.

$K^2 = -1$ *Anti de-Sitter space*; Einstein vacuum with cosmological constant -1, symmetry group $SO(2,3)$.

Put now $K = (0, \dots, 0, 1)$ then $w = 1$ and we obtain instead the hyperboloid

$$s^2 + t^2 = 1 + x^2 + y^2 + z^2, \tag{99}$$

As before for the Einstein cylinder, unwrap the time S^1 setting

$$(s, t) = \sec\psi(\cos\tau, \sin\tau), \qquad \tan^2\psi = x^2 + y^2 + z^2\,. \tag{100}$$

and this gives

$$\mathrm{d}s_{-1}^2 = \sec^2\psi\left(\mathrm{d}\tau^2 - \mathrm{d}s_{S^3}^2\right) \qquad \psi \in [0, \pi/2] \tag{101}$$

Thus we obtain the region $\psi \in [0, \pi/2]$ inside the Einstein cylinder.

Anti-de Sitter is important in the AdS/CFT correspondence.

A number of remarks are in order:

1. Infinity $\mathscr{I}$ is a null hypersurface for flat space, space-like for de Sitter, and time-like for AdS. We will see that the correlation with the sign of the cosmological constant is not a coincidence.

2. These last two representations as hyperboloids are in fact double covers of $\mathbb{M} - \{K \cdot X = 0\}$. In the $K^2 = 1, 0$ cases, $\mathscr{I}^-$ and $\mathscr{I}^+$ are identified in $\mathbb{M}$. This in particular shows that light cones of points of $\mathscr{I}^-$ refocus at the corresponding points of $\mathscr{I}^+$. We unwrap these spaces in order to avoid closed timelike curves.

3. The light cone of a point $X_0^I \in \mathbb{M}$ is the intersection of $X_0 \cdot X = 0$ with $\mathbb{M}$.

4. In the $K^2 = \pm 1$ cases we can still use the coordinates scaled so that $s + w = 1$ as we did for flat space with $K^2 = 0$ to obtain *Poincaré patch* coordinates

$$\mathrm{d}s_1^2 = \frac{\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2}{t^2}\,, \qquad \mathrm{d}s_{-1}^2 = \frac{\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2}{z^2} \tag{102}$$

   These have infinity at respectively $t = 0$ or $z = 0$ and taking $t > 0$ or $z > 0$, the patches miss out half the part of the space-times covering $\mathbb{M}$ (which in turn is double covered by the hyperboloids and so on). Sometimes one puts $t = \exp -T$ to obtain

$$ds_1^2 = dT^2 - \mathrm{e}^{2T}(dx^2 + dy^2 + dz^2)\,. \tag{103}$$

   The T is now the proper time of an observer at the origin in three space and emphasizes the exponential expansion seen by that observer; the coordinates cover the region in de Sitter space that can eventually be observed by this observer.

5. It is clear that Minkowski space and de Sitter are globally hyperbolic, but that AdS is not. For AdS, we need to present, not just data on an initial $t = const.$ hypersurface, but also data, or at least boundary conditions on the time-like infinity. Otherwise, one can imagine incoming radiation from infinity.

4.2 Cosmological models

The Friedmann-Robertson-Walker (FRW) models are models in which we assume a collection of comoving observers for whom the universe is homogeneous (the same for each observer) and isotropic (the same in every direction). It is straightforward to deduce from this that the space-time can divided up into spatial sections of constant curvature with 3-metrics $\mathrm{d}s^2_{E^3_k}$ are for $k = 0, 1, -1$ the flat 3-metric, the round sphere or hyperbolic 3-space respectively. The FRW metrics are

$$\mathrm{d}s^2_{FRW_k} = \mathrm{d}t^2 - a(t)^2 \,\mathrm{d}s^2_{E^3_k} \,. \tag{104}$$

They are all conformally flat and therefore can all be represented inside the Einstein cylinder. The value of k is determined by whether the density of the universe is greater than ($k = 1$), or less than ($k = -1$) or exactly equal to ($k = 0$) some critical value. This is currently too close to call.

The $k = +1$ case is the simplest since then

$$\mathrm{d}s^2_{FRW_1} = S(\tau)^2(\,\mathrm{d}\tau^2 - \,\mathrm{d}s^2_{S^3})\,, \qquad \tau(t) = \int^t \frac{dt}{a(t)}\,, \tag{105}$$

and $S(\tau(t)) = R(t)$. We usually assume perfect fluid energy momentum tensor

$$T_{ab} = \mathrm{diag}(\rho, p, p, p)\,,$$

In this conformal time coordinate, the main Einstein equation is the Friedmann equation is

$$\frac{da}{d\tau}^2 + ka^2 = \frac{8\pi G}{3}\rho a^4 + \frac{\lambda}{3}a^4\,. \tag{106}$$

Another independent equation can be expressed as the conservation equation

$$d\rho = -3(\rho + p)d\log a, . \tag{107}$$

We need an additional equation of state relating $p = f(\rho)$. For dust we have simply $p = 0$ and then the conservation gives simply $M = \rho a^3$ for some constant M (or for radiation $p = \rho/3$ we get $M = \rho a^4$).

The simplest 'dust' $k = 1$ case has, after solving the Friedmann equations

$$t = \frac{C}{2}(\tau - \sin\tau)\,, \qquad S(\tau) = \frac{C}{2}(1 - \cos\tau), \tag{108}$$

so we have a big bang $S = 0$ at $\tau = 0$ followed by a big crunch, again with $S = 0$ at $\tau = \pi$.

In the conformal diagram one sees cosmological horizons very clearly as light rays are at 45° on the Einstein cylinder. We see that in general an observer at later time can see far away regions A and B whose causal pasts do not intersect. This then makes the apparent homogeneity of the universe a surprise. In diagram that takes into account the size of a which tends to zero at the big bang, it is unclear whether the past of A and B can mix, whereas in the conformal diagram it is completely clear.

This surprising homogeneity is resolved by inflation which notes that there is a *surface of last scattering* at some time τ_s soon after the big bang before which we cannot see what was going on and for which the physics is not so clear. This surface is where radiation decouples from matter and so after this time, we can see what is going on, whereas before, we just have what we see from the cosmic microwave radiation. They then posit a period of inflation, modelled by gluing in an exponentially expanding region of de Sitter, which gives the pasts of A and B time to mix and homogenize so as to explain the apparent isotropy of the universe.

It is now known that the cosmological constant is positive. This now allows $a \to \infty$ for finite τ even with $k = 1$. With perfect fluid as above, when a is large, the λa^4 term dominates the RHS of the Friedmann equation, and in conformal time, for large a, the equation approximately gives

$$\frac{da}{d\tau} = \sqrt{\frac{\lambda}{3}} a^2 \quad a \sim \frac{1}{\tau_{\mathscr{I}} - \tau}$$

and $S(\tau)$ has a pole at $\tau = \tau_{\mathscr{I}}$ so that we get a $\mathscr{I}^+$ that looks like that of de Sitter.

For $k = 0, -1$ the result is similar. This exponential expansion arises because a positive cosmological constant looks like a stress-energy tensor $T_{ab} = \mathrm{diag}(\lambda, -\lambda, -\lambda, -\lambda)$, so that although the effective energy density is positive, the pressure is negative. At the current age of the universe, the contribution of λ to the energy density is thought to be of the same order as that of the matter including dark matter (visible matter being thought to be 3%, dark matter 30% and cosmological constant about 67% of the critical mass of the universe). Such a ratio of matter to cosmological constant is extremely high at early times, and extremely low at late times, and this sometimes leads to the 'why are we alive now?' question. The later periods are however, very cold and boring, and the early periods rather hot, and too early for structure to form, so there are anthropic arguments here.

The $k = 0, -1$ models can be obtained similarly as subsets of the Einstein cylinder (see for example Hawking and Ellis).

4.3 Conformal infinity in conformally curved spaces

Singularities are characterized by incomplete geodesics that cannot be extended beyond some finite time, perhaps because we have removed some region where the curvature is infinite. A space-time is nonsingular if it is *geodesically complete*, that is each geodesic can be extended to infinite affine parameter. Typical examples are isolated systems in which, perhaps some particles, fields or gravitational radiation come in from infinity, and interact without forming a black hole, and then escape again to infinity. In order to understand what is happening at large distances, we can introduce a concept of conformal infinity.

In curved space, we do not have a group of conformal symmetries, but we can nevertheless perform conformal rescalings $g \to \Omega^2 g$ and this leads to the following definition of conformal compactification in curved space.

Definition 4.1 *A conformal compactification of a space-time* (M, g) *is a manifold* $\tilde{M}$ *with boundary* $\mathscr{I} = \partial\tilde{M}$ *and metric* $\tilde{g}$ *such that*

1. $\tilde{g}$ *is smooth on* $\tilde{M}$
2. M *is diffeomorphic to the interior of* $\tilde{M}$,
3. *On* M *we have* $\tilde{g} = \Omega^2 g$ *with* Ω *smooth on* $\tilde{M}$, $\Omega \neq 0$ *on* M,
4. $\Omega = 0$, *and* $d\Omega \neq 0$ *on* $\mathscr{I} = \partial\tilde{M}$.

We can also specify the level of differentiability if desired.

We have already seen examples with Minkowski space, de Sitter space, AdS and so on. If M is globally hyperbolic we can see that $\mathscr{I} = \mathscr{I}^+ \cup \mathscr{I}^-$ where future infinity $\mathscr{I}^+$ is to the future of a Cauchy hypersurface and past infinity $\mathscr{I}^-$ to the past.

We have the theorem

Proposition 4.1 *Let (M, g) have conformal compactification $(\tilde{M}, \tilde{g})$, and suppose that the space-time asymptotically satisfies the Einstein equations with conformally invariant matter (so that the trace of the stress-energy tensor vanishes) with cosmological constant λ. Then $\mathscr{I}$ is space-like when $\lambda > 0$, time-like for $\lambda < 0$ and null when $\lambda = 0$.*

We furthermore have that if the trace-free Ricci tensor falls off fast enough at $\mathscr{I}$, then $\mathscr{I}$ is umbilic, i.e., the trace-free part of the extrinsic curvature vanishes. [The extrinsic curvature is $k_{ab} := \nabla_{(a}N_{b)}$ where N_a is the unit normal (and N^a is continued off the surface by $N^a\nabla_a N_b = 0$).] In the null case this implies that $\mathscr{I}$ is shear-free.

Proof: Define the *Schouten tensor*[6]

$$P_{ab} = -\frac{1}{2}R_{ab} + \frac{1}{12}Rg_{ab}\,. \tag{110}$$

This is constructed so that under a conformal rescaling we have

$$P_{ab} = \tilde{P}_{ab} + \Omega^{-1}\tilde{\nabla}_a\tilde{\nabla}_b\Omega - \Omega^{-2}\tilde{g}_{ab}\tilde{\nabla}_c\Omega\tilde{\nabla}^c\Omega \tag{111}$$

With the vanishing of the trace of the energy momentum tensor, we have in the physical metric $P_a^a = -R/6 = -\lambda/6$. So

$$-\frac{\lambda}{6} = P_a^a = \Omega^2\tilde{g}^{ab}P_{ab} = \Omega^2\tilde{P}_a^a + \Omega\tilde{\Box}\Omega - 4\tilde{\nabla}_a\Omega\tilde{\nabla}^a\Omega\,. \tag{112}$$

On $\mathscr{I}$, $\Omega = 0$ and so we have, defining the normal to $\mathscr{I}$ by $N_a = \tilde{\nabla}_a\Omega$

$$\tilde{g}^{ab}N_aN_b = \frac{\lambda}{24} \tag{113}$$

hence the first part of the proposition follows.

To obtain the second part we use the trace-free part of (111) to see that, multiplying through by Ω we have

$$(\tilde{\nabla}_aN_b - \frac{1}{4}\tilde{g}_{ab}\tilde{g}^{cd}\nabla_cN_d)|_{\Omega=0} = 0 \tag{114}$$

[6] This is defined in d-dimensions by

$$P_{ab} = -\frac{1}{d-2}\left(R_{ab} - \frac{1}{2(d-1)}R\right)\,, \tag{109}$$

with the same conformal transformation law. It plays a key role in conformal geometry. Note the sign flip relative to Riemannian definitions.

The extrinsic curvature is defined to be the projection of the covariant derivative of the normal into the surface and so this shows that its trace-free part vanishes. □

In the null case this implies that $\mathscr{I}$ has the intrinsic geometry of a light cone in Minkowski space in the sense that it is *shear free* in the sense of the Sachs equation.

4.4 Asymptotics and peeling

If the space-time is nonsingular, hence complete, we expect all light rays to make it to infinity both in the past and future, and if so, we say that the space-time is *asymptotically simple*. Such space-times can be thought of as perturbations of Minkowski space, de Sitter space or anti-de Sitter space. We have theorems now that tell us that such solutions can be constructed from generic but small data in some Sobolev norms. In the case of positive cosmological constant the stability of small perturbations of de Sitter was proved by Friedrich in the 1980s and for vanishing cosmological constant in the 1990s by Christodoulou and Klainerman and followers. However, in recent work, anti-de Sitter space has been shown to be unstable in this sense. For a start it is not globally hyperbolic, and one must impose some boundary conditions at $\mathscr{I}$ to obtain a well-posed initial value problem, and then these are usually chosen to be reflective so that waves can bounce back and forth leading to instabilities.

If the unphysical metric is smooth enough, we can also deduce that the Weyl tensor vanishes on $\mathscr{I}$. For zero cosmological constant it is possible to show that $\mathscr{I}$ has topology $S^2 \times \mathbb{R}$ and indeed this is typically also the case in black hole space-times with $\lambda = 0$. If so we can find, perhaps after a further rescaling, Bondi coordinates $(u, \zeta, \bar{\zeta})$ near $\mathscr{I}$ so that the unphysical metric is given by

$$\widetilde{ds}^2 = dud\Omega - \frac{d\zeta d\bar{\zeta}}{(1+|\zeta|^2)^2} + O(\Omega)\,, \tag{115}$$

where the second term is simply the sphere metric.

It is reasonable to expect conformally invariant and massless fields to continue smoothly to $\mathscr{I}$. Thus, if ϕ is a solution to the conformally invariant wave equation, $\tilde{\phi} = \phi/\Omega$ should be smooth on $\mathscr{I}$ in $(\tilde{M}, \tilde{g})$ at least if it is in the domain of dependence of M. When $\lambda > 0$, we can deduce that a linear massless field will evolve past $\mathscr{I}$ as if it wasn't there and so

$\phi_{A_1\ldots A_n}/\Omega = \phi^0_{A_1\ldots A_n}$ will be smooth and generically non-vanishing near $\mathscr{I}$ in the unphysical space-time, giving sharp asymptotic falloff of the physical field $\phi_{A_1\ldots A_n} = \Omega\phi^0_{A_1\ldots A_n}$. It is instructive to compare this falloff to that in terms of the affine parameter r along an outward going null geodesic. In the case when $\mathscr{I}$ is null, we find

$$\Omega \sim 1/r. \tag{116}$$

However, in the de Sitter case, it is easily seen that $\Omega \sim \exp(-t)$ when t is proper time along a time-like geodesic going out to $\mathscr{I}$ since $dt = f\tau/\tau$ where τ is the Einstein cylinder coordinate. When $\lambda = 0$, the situation is as before for the wave equation but more subtle for higher spin as different components of the spinor scale differently according to whether they are aligned with $\mathscr{I}$ or transverse. Taking o^A aligned along the null geodesic, we find that if ϕ_r is r contractions of ι and $n-r$ with o^A, then we have

$$\phi_r \sim \frac{1}{r^{n-r+1}}\,. \tag{117}$$

We can construct a spin-two field $\tilde{\psi}_{ABCD}$ on $\tilde{M}$ from the Weyl spinor Ψ_{ABCD} by defining

$$\tilde{\psi}_{ABCD} = \frac{\Psi_{ABCD}}{\Omega}\,. \tag{118}$$

The asymptotics above apply to this field, i.e., ψ_{ABCD} should be finite on $\mathscr{I}$. However, under the rescaling the Weyl tensor itself does not rescale. Thus we learn that the Weyl tensor itself should vanish on $\mathscr{I}$. The argument is more delicate when $\mathscr{I}$ is null, but follows when it has topology $S^2 \times \mathbb{R}$.

5 Black holes

More generally, we do not expect space-times to be complete and we expect singularities to form.

5.1 The Chandrasekhar limit

For a star whose nuclear fuel has burnt out, the pressure p is related to the density ρ by $P = \alpha\rho^\gamma$ for some constants α, γ.

$$\begin{aligned} \text{Gravitational potential energy} &\sim \frac{M^2}{R} \\ \text{Pressure energy} &\sim PV \sim PR^3 \sim \alpha\left(\frac{M}{R^3}\right)^\gamma R^3 \\ \text{Total energy} &\sim \alpha M^\gamma R^{3(1-\gamma)} - \frac{M}{R}\,. \end{aligned}$$

For $\gamma > 4/3$ a stable minimum exists for all M. For $\gamma < 4/3$ no stable minimum exists. The parameter γ measures the stiffness, and one can ask how stiff can matter become? The value $\gamma = 4/3$ value is in fact singled out by fermionic degeneracy pressure arising from the Pauli exclusion principle and represents a maximum stiffness.

For degenerate atoms/neutrons filling Fermi level p_F, the degeneracy implies that we have $n = \#/\text{vol} \sim p_F^3$ with one particle per cube of order of the wavelength. The density is then $\rho \sim m_n p_F^3$, where m_n is atom/neutron mass, pressure $\sim np_F \sim p_F^4$

$$P \sim m_n^{-4/3}\rho^{4/3}\,,$$

giving $\gamma = 4/3$. This implies

$$E = \frac{M^{4/3}}{R}(\alpha - M^{2/3})\,,$$

and so for $M > M_c = \alpha^{3/2}$ collapse is inevitable. According to the above $M_c \simeq 1/m_n^2 \sim$ one solar mass. This is the Chandrasekhar limit for white dwarfs (electron degeneracy) and Landau limit (neutron degeneracy) for Neutron stars.

These back-of the envelope calculations for the existence of black holes from 1930 are bolstered on the one hand by rigorous mathematical arguments in the form of the Hawking-Penrose singularity theorems from the 1960s, and more recently by ample observational evidence see Nasa website for examples.

The final state of gravitational collapse is understood to settle down to the Kerr or more simply the Schwarzschild solutions in which the star disappears inside a radius $R = 2M$, the Schwarzschild radius from which light can no longer escape.

5.2 Schwarzschild and the standard picture

The Schwarzschild metric

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - ds^2_{S^2} \tag{119}$$

provides the prototype nonrotating black hole exterior. It can be completed with an interior by gluing in a collapsing dust Friedman model

$$dt'^2 - R(t')^2(d\chi^2 + \sin^2\chi ds^2_{S^2}) \,. \tag{120}$$

We relegate this gluing to the exercises.

After collapse, it can be seen that the metric has issues at $r = 2m$ but these are resolved by use of the respectively retarded and advanced (Eddington-Finkelstein) coordinates u, v

$$du = dt - \frac{dr}{1 - \frac{2m}{r}} \,, \qquad dv = dt + \frac{dr}{1 - \frac{2m}{r}} \tag{121}$$

so that

$$(u, v) = (t - r_*, t + r_*) \,, \qquad r_* = r + 2m \log\left(\frac{r - 2m}{2m}\right) \,. \tag{122}$$

where r_* is the Regge-wheeler tortoise coordinate that places the horizon at $r_* = -\infty$. This allows us to put the metric in the form

$$ds^2 = \left(1 - \frac{2m}{r}\right) du^2 + 2dudr - r^2 ds^2_{S^2} \,, \tag{123}$$

and similarly with advanced coordinates, showing that there is no singularity at $r = 2m$. We see in fact that $r = 2m$ is a null hypersurface ruled by outgoing null geodesics, but the fact that $r = 2m$ means that the light rays are not escaping to infinity. It is an *event horizon*. For $r > 2m$, light rays with $\dot{r} > 0$ can and do escape. For $r < 2m$, all causal geodesics have future end point at $r = 0$.

These are the best coordinates for examining $\mathscr{I}^+$. The rescaling can be done with $\Omega = 1/r$ because r is an affine parameter on radial null geodesics. This yields unphysical metric

$$\widetilde{ds}^2 = \Omega^2 \left(1 - \frac{2m}{r}\right) du^2 + 2dud\Omega - ds^2_{S^2} \,, \tag{124}$$

and gives rise to the following picture:

In this picture it is clear that the event horizon satisfies the defining property

Definition 5.1 *The event horizon is the boundary of the past of $\mathscr{I}^+$.*

There is a corresponding time-reversed picture using coordinates (v, r, θ, ϕ). Using both we again have a problem at $r = 2m$

$$ds^2 = \left(1 - \frac{2m}{r}\right) dudv - r^2 ds^2_{S^2}\,, \tag{125}$$

but this can be used by using Kruskal coordinates

$$U = -\exp(-u/4m)\,, \qquad V = \exp(v/4m) \tag{126}$$

which yield

$$ds^2 = \frac{32m^3}{r} dUdV - r^2 ds^2_{S^2}\,, \tag{127}$$

and this now extends to negative values of U and V through $U = 0$ and $V = 0$ which give the event horizons since

$$UV = \left(1 - \frac{r}{2m}\right) \mathrm{e}^{r/2m}\,. \tag{128}$$

These give new asymptotic regions as $U, V \to -\infty$ and gives the full Kruskal extension with Penrose-Carter diagram:

We can see that the singularity $r = 0$ (which is a genuine curvature singularity) is a black hole to the future of every observer that crosses the future event horizon, or a white hole in the past. Time translation by

$$\partial_t = V\partial_V - U\partial_U \tag{129}$$

in this picture is much like a boost in $1 + 1$ dimensions.

Similar diagrams can be drawn for Reissner-Nordstrom, Kerr and the Kerr-Newman, see Hawking and Ellis although the latter have the novelty of *Cauchy horizons*, hypersurfaces beyond which neither fields nor space-time itself are determined by Cauchy data essentially as a consequence of *naked singularities*, singularities in the past of observers. However, these cannot be seen from infinity. These black hole solutions are unique subject to various assumptions (like the existence of a stationary Killing vector that looks like a time translation at large distances). They have extensions to versions with cosmological constant.

This final state is tightly constrained as in four dimensions we have powerful uniqueness theorems. Birkhoffs theorem says that any spherically symmetric vacuum solution is static, which then implies that it must be Schwarzschild. For Einstein-Maxwell system this extends to show that the only spherically symmetric solution is Reissner-Nordstrom. But suppose we know only that the metric exterior to a star is static. We further have:

Theorem 3 (Israel) *If (M,g) is an asymptotically-flat, static, vacuum spacetime that is non-singular on and outside an event horizon, then (M,g) is Schwarzschild.*

More remarkably we have

Theorem 4 (Carter-Robinson) *If (M,g) is an asymptotically-flat stationary and axi-symmetric vacuum spacetime that is non-singular on and outside an event horizon, then (M,g) is a member of the two-parameter Kerr family. The parameters are the mass M an the angular momentum J.*

The assumption of axi-symmetry has since been shown to be unnecessary by Hawking and Wald , i.e., for black holes, stationarity axisymmetry.

5.3 Horizons and black hole thermodynamics

For an asymptotically flat space-time, we define

Definition 5.2 *The* event horizon $\mathcal{H}$ *is the boundary of the past* $J^-(\mathscr{I}^+)$ *of* $\mathscr{I}^+$*, that is, it is the boundary of the region from which it is possible to escape to infinity along a causal curve.*

Much is known about event horizons under reasonable assumptions appropriate to isolated systems that settle down:

- $\mathcal{H}$ is a null hypersurface being the boundary of a past set (it clearly cannot be time-like as causal paths could then cross both ways, and if it were space-like there would be regions to its past that could not exit to $\mathscr{I}^+$).
- $\mathcal{H}$ is ruled (or foliated) by complete null geodesics.

- If $\mathscr{I}$ has topology $S^2 \times \mathbb{R}$, as appropriate for the exterior of an isolated system, then so does $\mathcal{H}$, with the $\mathbb{R}$ factor being the null geodesics.
- The cross-sectional area is bounded above.

This is a rather excessively global definition that requires knowledge of the whole space-time. One can also define with just local knowledge:

Definition 5.3 *a* closed trapped surface *is a two-surface of topology S^2 such that the outward pointing null geodesics have nonpositive expansion (i.e., the area will drop or be constant in any outward going null direction or $\rho \geq 0$ where ρ is the spin coefficient in the definition of the Sachs equation).*

Penrose's original singularity theorem deduces the existence of a singularity (in the form of geodesic incompleteness) from the existence of such a closed trapped surface. It is easy to see from the signs in the Sachs equations and following the outward going null geodesic normals off hte surface that a closed trapped surface leads to:

Definition 5.4 *an* apparent horizon *is a null hypersurface of topology $S^2 \times \mathbb{R}$ such that the expansion of the outward going null rays is nonpositive (i.e., the area is non-increasing to the future).*

The first of the Sachs equation for a null geodesic congruence generated by l gives

$$\nabla_l \rho = \rho^2 + \sigma\bar{\sigma} + \Phi_{00} \geq \rho^2 \,. \tag{130}$$

Thus if $\rho \geq 0$ then it cannot decrease. (Recall that if A is the area element, $\mathcal{L}_l A = -2\rho A$.)

However, Penrose's theorem doesnt deduce the location of the singularity! In particular it is not clear that an apparent horizon is hidden inside an event horizon and the following is open:

The cosmic censorship hypothesis: All singularities that arise from evolving from an initial data hypersurface are hidden behind an event horizon and so cannot be seen from infinity.

Generally speaking we assume that a black hole settles down to being stationary or static. Then, the event horizon must settle down to a null hypersurface with finite cross sectional area (otherwise geodesics will be escaping to infinity). Once a black hole horizon settles down, its area is constant. Assuming that the black hole is becoming stationary or static, it follows that it is (under suitable analyticity assumptions) a *Killing horizon*:

Definition 5.5 *A Killing horizon is a null hypersurface on which a Killing vector k_a becomes null, so that the surface is defined by $k_a k^a = 0$ and $k^a \neq 0$. Thus k^a is tangent to the null geodesic generators of the horizon.*

The fact that k^a is Killing means that we must have $\rho = \sigma = 0$.

It follows from the black hole uniqueness theorems that, even if we started from some collapse scenario, the final black hole, if essentially static or stationary, is Kerr Newman or Schwarzschild. These all have a similar structure to Schwarzschild in that they can be continued analytically back to a point where the standard future event horizon intersects a past one at a 2-surface C.

Definition 5.6 *Such a Killing Horizon is said to be a bifurcate Killing horizon if there exists cross-section C of topology S^2 on which k^a vanishes—it is bifurcate because then in a neighbourhood of there is a transverse horizon such that $k^a = U\partial_U - V\partial_V$ as for the crossover in Schwarzschild.*

On a Killing horizon we can define

Definition 5.7 *The* surface gravity κ *is defined by*

$$\nabla_a k_b k^b = -2\kappa k_a \,, \qquad \text{or } k^a \nabla_a k^b = \kappa k^b \,, \tag{131}$$

where in the static case, k_a is understood to be normalized to have $k_a k^a = 1$ at large distances. For Schwarzschild $\kappa = 1/4m$.

Black hole thermodynamics starts with the Bekenstein bound on the entropy S: in a region of radius R and mass-energy E the entropy is constrained by

$$S < \frac{2\pi k R E}{\hbar c} \tag{132}$$

where we have not set the usual fundamental constants $k, \hbar, c$ to unity. It was arrived at by consideration of throwing objects with entropy into black holes and trying to avoid violations of the second law arising from the black hole eating entropy. In this view, the black does have entropy

$$S_{BH} = \frac{kA}{4G} \tag{133}$$

and this is taken to be the maximal entropy state, i.e., the Bekenstein bound is saturated by the black hole entropy.

Classically, one does not think of black holes as having microstates that could give rise to an entropy in view of the black hole uniqueness theorems. These seem to imply that the black hole state is unique, whereas an enropy suggests the existence of many equally likely microstates compatible with given macroscopic observables. The black hole entropy is usually understood as having its origin in quantum gravity.

This chain of reasoning subsequently led to the *Holographic principle*, that the maximum number N of states in a spatial region of radius R satisfies

$$N < \exp S_{BH}(R) \tag{134}$$

This comes from the definition of entropy of a system as $S := -\sum_i p_i \log p_i$ where p_i is the probability of the ith state. If the system is equidistributed, $p_i = 1/N$, where N is the number of states (the dimension of the Hilbert space of the system) we obtain $S = \log N$. This is counter-intuitive without general relativity because one thinks of the number of states in a region as being the exponential of the volume rather than the area. However, gravitational collapse reduces this if there is too much matter (too many particles) and indeed the vast bulk of the entropy is undertood to be gravitational.

The second law of thermodynamics: if the entropy is equated with the area of the event horizon in black hole thermodynamics, the 2nd law states that it can only increase. We have the area theorem

Proposition 5.1 *The area of an event horizon is non-decreasing.*

Proof: This is a simple consequence of the Sachs (or Raychauduri) equations

$$\dot{\rho} = \rho^2 + |\sigma|^2 + \Phi_{00}\,. \tag{135}$$

This shows that in particular $\dot{\rho} \geq \rho^2$ when the dominant energy condition is satisfied. Thus, if $\rho = \rho_0 > 0$ at some affine parameter value $t = 0$ on the generator, it is bounded below by the solution

$$\rho_0(t) = \frac{\rho_0}{1 - \rho_0 t}\,, \tag{136}$$

the solution to $\dot{\rho} = \rho^2$ with the same initial condition. Thus $\rho \to \infty$ in finite time. This introduces a *cusp* after which the null geodesic must then leave the horizon (see picture), contradicting its being a generator of $\mathcal{H}$. $\Box$

The first law of black hole thermodynamics: for a variation of a closed system with rotation and charge can be stated as

$$dE = TdS + \Omega dJ + \Phi_H dQ \tag{137}$$

Here E is the total energy, T the temerature, Ω the angular velocity, J the angular momentum, Q the charge and ϕ the electrostatic potential. In the context of black holes, the total energy is the mass, we identify the temperature with the surface gravity by

$$T = \kappa/2\pi \tag{138}$$

and S with the area.

For Reissner Nordstrom, $\Phi = \Phi_H$ is the electric potential at the horizon and Q the total charge, and, in the case of the Kerr solution, Ω is the angular velocity, and J the angular momentum.

There are a number of strategies for proving these formulae. The most basic is to simply establish sufficient relations between the various quantities $(M, A, \Omega_H, J, \phi_H, Q)$ as can be read off from the black hole metric, and then to differentiate it. The simplest example is for Schwarschild where the area is that associated with the Schwarzschild radius $r = 2M$, $A = \pi 4M^2$, upon which differentiation yields

$$dM = \frac{dA}{8\pi M} = \frac{\kappa}{2\pi} dA\,, \tag{139}$$

giving the most basic version.

If we wish to introduce charge, we must consider the Reissner-Nordstrom solution

$$ds^2 = \frac{\Delta(r)}{r^2} dt^2 - \frac{r^2}{\Delta(r)} dr^2 - r^2 ds^2_{S^2}\,, \qquad \Delta(r) = r^2 - 2Mr + Q^2. \tag{140}$$

This satisfies the Einstein equations with electromagnetic potential

$$A = \frac{Q}{r} dt\,. \tag{141}$$

The Killing horizons are where $\Delta = 0$ giving

$$r_\pm = M \pm \sqrt{M^2 - Q^2}\,, \tag{142}$$

assuming $Q < M$. The outer one is the event horizon and a short calculation shows that differentiating the obvious relation $A = \pi r_+^2$ now gives

$$dM = \frac{\sqrt{M^2 - Q^2}}{2\pi r_+^2} dA + \frac{Q}{r_+} dQ \,. \tag{143}$$

The coefficient of dQ is indeed the value of the potential at the horizon. It is a more complicated task to see that the surface gravity does indeed appropriately give the coefficient of dA (see the exercises). Even more nontrivially, this works as stated above for the Kerr-Newman solution where there is also rotation.

The zeroth law of Black hole thermodynamics: In the analogy with thermodynamics, κ plays the role of temperature via $T = \kappa/2\pi$. The zeroth law is that the temperature is constant in equilibrium. It is easy to see that the surface gravity κ is constant up the generators of $\mathcal{H}$, because k^a is Killing. We will see in the problems that for a bifurcate Killing horizon κ is actually constant over the horizon. Hence it is constant everywhere. The next result follows in greater generality but we will not prove it here.

There is also a third law, that the entropy of an object at absolute zero is zero. This fails for black holes for a number of reasons, but a vaguer version, that once cannot approach absolute zero temperature with a finite number of processes does seem reasonable, as $T \to 0$ corresponds to $M \to \infty$.

The glaring omission in all this is of course that the temperature of a black hole classically would seem to have to be zero. This will be seen to be resolved by Hawking radiation.