Topological groups, 2021–2022

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Course overview

Groups like the integers, the circle, and general linear groups (over $\mathbb{R}$ or $\mathbb{C}$) share a number of properties naturally captured by the notion of a topological group. Providing a unified framework for these groups and properties was an important achievement of 20th century mathematics, and in this course we shall develop this framework.

Highlights will include the existence and uniqueness of Haar integrals for locally compact topological groups, the topology of dual groups, and the existence of characters in various topological groups. Throughout, the course will use the tools of analysis to tie together the topology and algebra, getting at superficially more algebraic facts by analytic means.

Course synopsis

[6 lectures] Definition of topological groups. Examples and non-examples. Quotient groups. Subgroups. Compactness and local compactness. Non-functional separation axioms. The Open Mapping Theorem.
[5 lectures] Complete regularity of topological groups. Continuous partitions of unity and Fubini's Theorem. Existence and uniqueness of Haar integrals.
[5 lectures] Peter-Weyl Theorem for compact topological groups. Dual groups of topological groups. Local compactness of the dual of a locally compact topological group.

References

There are other notes on similar topics with a slightly different focus: [Fol95, Kör08, Kra17, Meg17] and [Rud90].

General prerequisites

The course is designed to be pretty self-contained. We assume basic familiarity with groups as covered in Prelims Groups and Group Actions (see *e.g.* [Ear14]). We shall also assume familiarity with Prelims Linear Algebra (see *e.g.* [May20]) and Part A: Metric Spaces and Complex Analysis (see *e.g.* [McG19]) for material on metric and normed spaces.

Familiarity with topology is essential, though not much is required. What we use (and more) is covered in Part A: Topology (see *e.g.* [DL18]), with the exception of Tychonoff's Theorem. This can be informally summarised as saying that a non-empty product of compact spaces is compact, and there is no harm in taking it as a black box for the course. Those interested in more detail may wish to consult Part C: Analytic Topology (see *e.g.* [Kni18]).

The Axiom of Choice is sometimes formulated as saying that an arbitrary product of non-empty sets is non-empty, and in this formulation it may be less surprising that it can be used to prove Tychonoff's Theorem. It turns out that the converse is also true, *i.e.* Tychonoff's Theorem (and the other axioms of set theory) can be used to prove the Axiom of Choice[1].

Finally no familiarity with functional analysis is assumed, though there are clear similarities and parallels for those who do have some. See *e.g.* [Pri17] and [Whi19].

Teaching

A first draft of these notes is on the website, but they will be updated after each lecture with any resulting changes. This document was compiled on 3rd May, 2022 at 10:13.

Lectures will be supplemented by some tutorial-style teaching where we can discuss the course and also exercises from the sheets. Once I have a list of the MFoCS students attending I shall be in touch to arrange these.

Contact details and feedback

Contact tom.sanders@maths.ox.ac.uk if you have any questions or feedback.

[1] Those unfamiliar and looking for a reference may wish to consult the notes [Ter10].

1 Groups with topologies

We say a group G is **written multiplicatively** to mean that the binary operation of the group is denoted $G^2 \to G; (x, y) \mapsto xy$; with inversion denoted $G \to G; x \mapsto x^{-1}$; and identity denoted 1_G. If G is *Abelian* then we say it is **written additively** to mean that the binary operation of the group is denoted $G^2 \to G; (x, y) \mapsto x + y$ and called **addition**; with inversion denoted $G \to G; x \mapsto -x$ and called **negation**; and identity denoted 0_G. ⚠ All groups written additively will be Abelian, but not all Abelian groups will be written additively.

A group G that is also a topological space is called a **topologized group**. Without any additional assumptions these are no more than their constituent parts: a group and a topological space. When the group inversion $G \to G$ and the group operation $G^2 \to G$ are both continuous, where in the latter case G^2 has the product topology on G^2, we say G is a **topological group**.

Example 1.1 (Indiscrete groups)**.** Any group G endowed with the indiscrete topology is a topological group since any map into an indiscrete space is continuous.

Example 1.2 (Discrete groups)**.** Any group G endowed with the discrete topology is a topological group since the product of two copies of the discrete topology is discrete – so both the topological spaces G and G^2 are discrete – and any map from a discrete space is continuous.

The reals under addition may be endowed with the discrete or indiscrete topologies to make them into a topological group as above. However, neither of these is the 'usual' topology on $\mathbb{R}$ which is generated by intervals without their endpoints.

Example 1.3 (The real line)**.** The group $\mathbb{R}$ (the operation is addition) endowed with its usual topology is a topological group. The reals are a metric space and so the topology is completely determined by sequences. Hence the relevant continuity is just the algebra of limits: in particular, if $x_n \to x_0$ then $-(x_n) = (-1)x_n \to (-1)x_0 = -x_0$; and if additionally $y_n \to y_0$, then $x_n + y_n \to x_0 + y_0$.

Example 1.4 (Normed spaces)**.** The additive group of a normed space X with the topology induced by the norm is a topological group by essentially the same argument as in Example 1.3 since addition and scalar multiplication are continuous in the norm. In particular, $\mathbb{R}^n$ and $\mathbb{C}^n$ are topological groups under addition.

Example 1.5. The non-zero complex numbers, $\mathbb{C}^*$, form a multiplicative group and with the usual topology this is a topological group by the algebra of limits again: if $x_n \to x_0$ in $\mathbb{C}^*$ then $x_n^{-1} \to x_0^{-1}$; and if additionally $y_n \to y_0$ then $x_n y_n \to x_0 y_0$.

There are more examples in Proposition 1.54 and on the exercise sheets in Exercises I.7, II.2 & III.3, and we shall see later in Propositions 1.27, 2.9 & 2.15, that subgroups, product groups, and quotient groups are naturally topological groups when the underlying groups are topological groups and these constructions can be used to generate yet more examples.

Group notation

Remark 1.6. Suppose that G is a group written multiplicative and $S, T \subset G$. We write

$$S^{-1} := \{s^{-1} : s \in S\} \text{ and } ST := \{st : s \in S, t \in T\}.$$

For $n \in \mathbb{N}_0$ we define S^n inductively by $S^0 := \{1_G\}$ and $S^{n+1} := S^n S$, and $S^{-n} := (S^{-1})^n$. It will also be convenient to write $xS := \{x\}S$ and $Sx := S\{x\}$ for $x \in G$.

If G is written additively then the above notation changes in the obvious way so we write $S + T$ instead of ST *etc.*

Remark 1.7. ⚠ In general $SS^{-1} \neq S^0$ and $S^2 \neq \{s^2 : s \in S\}$.

Remark 1.8. ⚠ G^n denotes the n-fold Cartesian product $G \times \cdots \times G$ *not* the product defined in Remark 1.6; the product is just G.

We say $S \subset G$ is **symmetric** if $S = S^{-1}$.

Remark 1.9. If S and T are symmetric then $S \cap T$ is symmetric.

Remark 1.10. We write $\langle S \rangle$ for the group generated by S, that is $\bigcap \{H \leqslant G : S \subset H\}$, the intersection of all the subgroups of G containing S.

Remark 1.11. If S is symmetric then $\langle S \rangle = \bigcup_{n \in \mathbb{N}_0} S^n$ by the subgroup test.

Semitopological, quasitopological, and paratopological groups

Suppose that G is a topologized group written multiplicative. We say that the group operation on G is **separately continuous** if the maps $G \to G; x \mapsto xy$ and $G \to G; x \mapsto yx$ are continuous for all $y \in G$.

Remark 1.12. The maps $G \to G^2; x \mapsto (x, y)$ (and $G \to G^2; x \mapsto (y, x)$) are continuous for all $y \in G$ and so is the group multiplication is continuous then is is separately continuous. Sometimes we say that the group operation is **jointly continuous** when it is continuous to emphasise the difference with separate continuity.

Remark 1.13. Separate continuity of the group operation is exactly equivalent to saying that xU and Ux are open (resp. closed) whenever U is open (resp. closed) and $x \in G$.

A topologized group G in which the group operation is separately continuous is called a **semitopological group**. If additionally inversion is continuous then we call it a **quasitopological group**. If the group operation is jointly continuous (but nothing is assumed about inversion) then we call G a **paratopological group**.

Our purpose in introducing these structures is to understand exactly which topological hypothesis lead to which conclusions in topological groups, but they are also studied in their own right. For a much more detailed development including many examples and open problems see [AT08, Chapters 1 & 2].

Remark 1.14. In view of Remark 1.12, we have the following implications:

$$\begin{array}{ccc} & \text{Topological} & \\ \swarrow & & \searrow \\ \text{Quasitopological} & & \text{Paratopological} \\ \searrow & & \swarrow \\ & \text{Semitopological} & \end{array}$$

Example 1.17 gives a semitopological group that is neither quasitopological nor paratopological; Example 1.15 gives a paratopological group that is not topological; and Example 1.16 gives a quasitopological group that is not topological. So none of the implications can be reversed.

Since a quasitopological group that is also a paratopological group is a topological group these examples also show that there can be no implication (in either direction) between the properties of being quasitopological and paratopological.

Example 1.15 (Reals with the right order topology)**.** The set $\{(a, \infty) : a \in \mathbb{R}\} \cup \{\varnothing, \mathbb{R}\}$ is a topology on $\mathbb{R}$, and $\mathbb{R}$ with the operation of addition is a paratopological group since

$$\{(x, y) : x + y \in (a, \infty)\} = \bigcup_{b \in \mathbb{R}} (a - b, \infty) \times (b, \infty)$$

so that the preimage of the open set (a, ∞) is open in the product topology. This paratopological group is *not* a topological group since $(-\infty, -a)$ is *not* open (for any $a \in \mathbb{R}$).

Example 1.16 (Groups with cofinite topologies)**.** Since intersections and finite unions of finite sets are finite, any set may be equipped with a topology in which the *proper* closed sets are the finite sets – this is called the **cofinite topology**.

A group G equipped with the cofinite topology is a quasitopological group since U^{-1} is finite if U is finite (so inversion is continuous), and xU and Ux are finite if U is finite (so multiplication is separately continuous).

If G is finite then the cofinite topology is the same as the discrete topology and G is a topological group (as in Example 1.2). On the other hand, if G is infinite we shall see in Remark 1.33 that it is *not* a topological group.

Example 1.17. Since intersections and finite unions of countable subsets of $\mathbb{R}$ that are bounded below are, themselves, countable subsets of $\mathbb{R}$ that are bounded below, $\mathbb{R}$ may be equipped with a topology in which the proper closed sets are the countable subsets of $\mathbb{R}$ that are bounded below.

A translate of a set that is countable and bounded below is still countable and bounded below, and hence $\mathbb{R}$ equipped with this topology is a semitopological group.

On the other hand, $\mathbb{R}\backslash\mathbb{N}_0$ is open, but $-(\mathbb{R}\backslash\mathbb{N}_0) = \mathbb{R}\backslash(-\mathbb{N}_0)$ is not so inversion is not continuous, and so this is not a quasitopological group. It is not a paratopological group either as we shall see in Remark 1.33, which are essentially the same reasons as in Example 1.16.

There are a few key lemmas (Lemmas 1.18, 1.22, 1.24,1.29, 1.31, and 1.35) which we highlight in red because they each capture a crucial technique or idea.

Lemma 1.18 (Key Lemma I)**.** *Suppose that G is a topologized group in which inversion is continuous. If U is a neighbourhood of 1_G then U contains a symmetric open neighbourhood of the identity; if K is a compact set then K is contained in a compact symmetric set; and if S is symmetric then $\overline{S}$ is symmetric*

Proof. If U is a neighbourhood of 1_G then U contains an open neighbourhood V of 1_G. Put $S := V \cap V^{-1}$ which is open and contains 1_G (since $1_G^{-1} = 1_G$) and moreover $S = S^{-1}$ so that S is a symmetric open neighbourhood of 1_G contained in U.

Since inversion is continuous and K is compact, the image of K, K^{-1}, is compact and since the union of compact sets is compact we conclude that $K \cup K^{-1}$ is a compact symmetric set.

Finally, inversion is continuous and so the preimage of $\overline{S}$ under inversion (which is the same as the image of $\overline{S}$ under inversion) is the set $\overline{S}^{-1}$ and is closed and contains $S^{-1} = S$. It follows that $\overline{S} \subset \overline{S}^{-1}$. But $\overline{S}^{-1} \subset (\overline{S}^{-1})^{-1} = \overline{S}$, and we conclude that $\overline{S}^{-1} = \overline{S}$. □

Remark 1.19. In particular Lemma 1.18 applies to quasitopological groups.

Remark 1.20. If K is compact and $1_G \in K$ then there is a symmetric compact $C \subset K$ with $1_G \in C$, namely $C = \{1_G\}$. ⚠Intersections of compact sets in topological groups are not necessarily compact. See Exercise I.4.

Example 1.21. The only sets in $\mathbb{R}$ with the right order topology (Example 1.15) that are symmetric and open are $\emptyset$ and $\mathbb{R}$. Hence $(-1, \infty)$ is a neighbourhood of the identity that does not contain a symmetric neighbourhood of the identity; $[1, \infty)$ is compact, but $(-\infty, -1] \cup [1, \infty)$ is not compact; and $\overline{\{1, -1\}} = (-\infty, 1]$ which is not symmetric despite $\{1, -1\}$ being symmetric. In particular every conclusion of Lemma 1.18 may fail if 'topologized group with continuous inverse' is replaced by 'paratopological group'.

Lemma 1.22 (Key Lemma II)**.** *Suppose that G is a semitopological group, U is open and V is any set. Then UV and VU are open, and U is a neighbourhood of x if and only if $x^{-1}U$ (or Ux^{-1}) is a neighbourhood of the identity.*

Proof. First, $UV = \bigcup_{v \in V} Uv$ which is a union of open sets by the first part and hence open. Similarly VU is a union of open sets and so open. Finally, if U is a neighbourhood of x then there is an open set $U_x \subset U$ containing x. Hence $x^{-1}U_x$ is an open set containing 1_G and contained in $x^{-1}U$, which is to say $x^{-1}U$ is a neighbourhood of the identity. Similarly if $x^{-1}U$ is a neighbourhood of the identity then U is a neighbourhood of x, and the same two arguments also work for Ux^{-1}. $\square$

A topological space X is **Fréchet** if every singleton in X is closed.

Lemma 1.23. *Suppose that G is a semitopological group. Then G is Fréchet if and only if $\{1_G\}$ is closed.*

Proof. This follows since $\{x\} = x\{1_G\}$ is closed if and only if $\{1_G\}$ is closed – see Remark 1.13. $\square$

Lemma 1.24 (Key Lemma III)**.** *Suppose that G is a semitopological group, S is a set and V is an open neighbourhood of the identity. Then $\overline{SV} \subset SVV^{-1}$.*

Proof. Let $A := G\backslash(SVV^{-1})$ and $B := G\backslash(AV)$. B is closed since AV is open by Lemma 1.22. If $x \in SV$ and $x \in AV$ then there is some $v \in V$ such that $xv^{-1} \in A$, so $xv^{-1} \notin SVV^{-1}$, a contradiction. Hence $SV \subset B$ and since B is closed $\overline{SV} \subset B$. Now if $x \in B$ then by definition $x \notin AV$ and so in particular $x \notin A$ (since $1_G \in V$) and hence $x \in SVV^{-1}$ as claimed. $\square$

The next result is, perhaps, a little surprising.

Corollary 1.25. *Suppose that G is a semitopological group and $H \leqslant G$. If H is a neighbourhood in G then H is open in G; and if H is open in G then H is closed in G.*

Proof. If H is a neighbourhood of some $x \in G$ then by Lemma 1.22 there is an open set U such that $x^{-1}U$ is an open set containing the identity. Now $H = HU$ is open, again by Lemma 1.22.

For the second part, if H is open then by Lemma 1.24 $\overline{H} \subset HH^{-1} = H$ and so H is closed. $\square$

Remark 1.26. If U is a neighbourhood in a semitopological group G then by Corollary 1.25 $\langle U \rangle$ is closed so $\overline{U} \subset \langle U \rangle$ and hence $\langle \overline{U} \rangle = \langle U \rangle$. ⚠This need not be true if U is not a neighbourhood, for example $\mathbb{Q}$ in $\mathbb{R}$ with its usual topology, has closure equal to $\mathbb{R}$, but $\langle \mathbb{Q} \rangle$ is countable and so does not contain $\mathbb{R}$.

Proposition 1.27. *Suppose that G is a topologized group and $H \leqslant G$ is given the subspace topology. If group inversion on G is continuous, then it is continuous on H; if multiplication is separately continuous on G, then it is separately continuous on H; and if multiplication is jointly continuous on G then it is jointly continuous on H. In particular if G is a topological (resp. paratopological, quasitopological, or semitopological) group then so is H.*

Proof. Suppose U is an open set in H, and let W be an open subset of G such that $U = W \cap H$. Then $U^{-1} = (W \cap H)^{-1} = W^{-1} \cap H^{-1} = W^{-1} \cap H$, but W^{-1} is open in G and so U^{-1} is open in H *i.e.* inversion is continuous; similarly, for $x \in H$, $xU = x(W \cap H) = (xW) \cap H$, but xW is open in G and so xU is open in H (and similarly for Ux), so multiplication is separately continuous.

For joint continuity of multiplication, let $V := \{(x, y) \in G^2 : xy \in W\}$ so that $V \cap H^2 = \{(x, y) \in H^2 : xy \in U\}$. Since multiplication on G is jointly continuous, by definition of the product topology there is a set $\mathcal{S}$ of products of open sets in G such that

$$V = \bigcup \{S \times T : S \times T \in \mathcal{S}\}.$$

Now $(S \times T) \cap H^2 = (S \cap H) \times (T \cap H)$, and so the preimage of U under multiplication on H is open in the product of the subspace topology on H with itself. That is to say, multiplication is jointly continuous on H and the result is proved. □

Example 1.28. $S^1 := \{z \in \mathbb{C}^* : |z| = 1\}$ is a subgroup of $\mathbb{C}^*$ and so it is a topological group. In this case it is closed, but in general we are not making the assumption that any subgroups we are considering are (topologically) closed.

We now turn to a couple of key lemmas which (like Proposition 1.27) make essential use of *joint* continuity.

Lemma 1.29 (Key Lemma IV)**.** *Suppose that G is a paratopological group and $K_1, \dots, K_n$ are compact subsets of G. Then $K_1 \cdots K_n$ is compact. In particular, if K is compact then K^n is compact for all[2] $n \in \mathbb{N}_0$.*

Proof. The (topological) product of two compact sets is compact so if $K_1 \cdots K_{n-1}$ is compact and K_n is compact then $(K_1 \cdots K_{n-1}) \times K_n$ is compact. But then the continuous image of a compact set is compact and so $K_1 \cdots K_n = (K_1 \cdots K_{n-1})K_n$ is compact and the result follows by induction on n. □

Remark 1.30. Exercise I.2 gives an example of a quasitopological group where the conclusion above does not hold.

[2] Note that $K^0 = \{1_G\}$ by definition and so is compact since it is finite.

Lemma 1.31 (Key Lemma V)**.** *Suppose that G is a paratopological group and X is a neighbourhood of z. Then there is an open neighbourhood of the identity V such that $zV^2 \subset X$. Moreover, if G is a topological group then V may be taken to be symmetric.*

Proof. Let $U \subset X$ be an open neighbourhood of z. The map $(x, y) \mapsto xy$ is continuous and so $\{(x, y) : xy \in U\}$ is an open subset of $G \times G$. By definition of the product topology there is a set $\mathcal{S}$ of products of open sets in G such that

$$\{(x, y) : xy \in U\} = \bigcup \{S \times T : S \times T \in \mathcal{S}\}.$$

Since $z1_G = z \in U$, there is some $S \times T \in \mathcal{S}$ such that $(z, 1_G) \in S \times T$. Thus S is an open neighbourhood of z and T is an open neighbourhood of the identity, so by Lemma 1.22 $V := (z^{-1}S) \cap T$ is an open neighbourhood of the identity. Now $zV \subset S$ and $V \subset T$ and so $zV^2 \subset U$ as required. Moreover, if G is a topological group so inversion is also continuous then by Lemma 1.18 V contains a symmetric open neighbourhood of the identity, and the conclusion follows by nesting. □

Example 1.32. Suppose G is a Fréchet semitopological group all of whose proper closed sets are smaller than any of its non-empty open sets, meaning there is no injection from a non-empty open set to a proper closed set. We claim G cannot satisfy the conclusion of Lemma 1.31:

First, $G \backslash \{1_G\}$ is open and non-empty *i.e.* there is some $z \in G$. If G satisfied the conclusion of Lemma 1.31 then there would be a non-empty open set $V \subset G$ such that $zV^2 \subset G \backslash \{1_G\}$. The map $V \to G; v \mapsto v^{-1}z^{-1}$ maps into the proper closed set $G \backslash V$ since $1_G \notin zV^2$, and is an injection since G is a group. This contradicts our assumption and the claim is proved.

Remark 1.33. Example 1.16 shows that a group with the cofinite topology is a quasitopological group, and since singletons are finite and so closed it is Fréchet. If the group is infinite then this quasitopological group additionally satisfies the hypotheses of Example 1.32 since the proper closed sets are finite, while the non-empty open sets are infinite. It follows that such a group is not a paratopological group, and hence not a topological group as claimed in Example 1.16.

Example 1.32 also applies to the semitopological group of reals endowed with the topology from Example 1.17, since every singleton is closed (so the topology is Fréchet) and every proper closed set there is countable, while every non-empty open set is uncountable. This shows that they do not enjoy the conclusion of Lemma 1.31 and hence are not a paratopological group as claimed in Example 1.17.

Lemma 1.31 can be used to establish some uniformity in open covers of compact sets. A cover $\mathcal{U}$ is a **refinement** of a cover $\mathcal{V}$ of a set X if $\mathcal{U}$ is a cover of X and each set in $\mathcal{U}$ is contained in some set in $\mathcal{V}$.

Remark 1.34. Refinements are transitive meaning that if $\mathcal{W}$ is a refinement of $\mathcal{V}$ and $\mathcal{V}$ is a refinement of $\mathcal{U}$ then $\mathcal{W}$ is a refinement of $\mathcal{U}$.

Lemma 1.35 (Key Lemma VI)**.** *Suppose that G is a paratopological group and $K \subset G^n$ is compact for some $n \in \mathbb{N}$, and $\mathcal{U}$ is an open cover of K. Then there is an open neighbourhood of the identity $U \subset G$ such that $\{x_1U \times \cdots \times x_nU : x \in K\}$ is a refinement of $\mathcal{U}$. If G is a topological group then U may be taken to be symmetric.*

Proof. First, the structure of the product topology (and Lemma 1.22) means that we can pass to a refinement of $\mathcal{U}$ where for each $x \in K$ there are open neighbourhoods of the identity $U_1^{(x)}, \dots, U_n^{(x)}$ (our notation is a little clumsy here to make the x-dependence explicit) such that $x_1U_1^{(x)} \times \cdots \times x_nU_n^{(x)}$ is in the refinement. The set $\bigcap_{i=1}^n U_i^{(x)}$ is an open neighbourhood of the identity and so by Lemma 1.31 there is a (symmetric if G is topological) open neighbourhood of the identity U_x such that $U_x^2 \subset U_i^{(x)}$ for all $1 \leqslant i \leqslant n$. In particular, $\mathcal{V} := \{x_1U_x \times \cdots \times x_nU_x : x \in K\}$ is an open cover of K and a refinement of $\mathcal{U}$.

By compactness of K there is a finite set $F \subset K$ such that $\mathcal{W} := \{x'_1U_{x'} \times \cdots \times x'_nU_{x'} : x' \in F\}$ is a cover of K. Let $U := \bigcap_{x' \in F} U_{x'}$ which is a finite intersection of (symmetric if G is topological) open neighbourhoods of the identity and so a (symmetric if G is topological) open neighbourhood of the identity. Since $\mathcal{W}$ is a cover of K, for each $x \in K$ there is some $x' \in F$ such that $x \in x'_1U_{x'} \times \cdots \times x'_nU_{x'}$, and hence

$$\begin{aligned} x_1U \times \cdots \times x_nU &\subset x'_1U_{x'}U \times \cdots \times x'_nU_{x'}U \\ &\subset x'_1U_{x'}^2 \times \cdots \times x'_nU_{x'}^2 \subset x'_1U_1^{(x')} \times \cdots \times x'_nU_n^{(x')} \end{aligned}$$

so that $\{x_1U \times \cdots \times x_nU : x \in K\}$ is a refinement of $\mathcal{V}$ which in turn is a refinement of $\mathcal{U}$ as required. □

Remark 1.36. The lemma above is not unrelated to the Generalised Tube Lemma from topology (see *e.g.* [Mun00, Lemma 26.8]), which is also known as Wallace's Theorem.

This proposition highlights an important interplay between compactness and the group structure, and has content even in seemingly simple cases:

Corollary 1.37. *Suppose that G is a topological group, A is a compact set and B is an open set containing A. Then there is a symmetric open neighbourhood of the identity U such that $\overline{AU} \subset B$. In particular, every neighbourhood of x contains a closed neighbourhood of x.*

Proof. Apply Lemma 1.35 with $n = 1$ to the open cover $\{B\}$ of A to get an open neighbourhood of the identity, V, such that $AV \subset B$. By Lemma 1.31 there is a symmetric open neighbourhood of the identity U such that $UU^{-1} = U^2 \subset V$, and so by Lemma 1.24 $\overline{AU} \subset AUU^{-1} \subset AV \subset B$ as required.

The last part follows immediately since the given neighbourhood contains an open neighbourhood B of x. The set $\{x\}$ is compact and so there is an open neighbourhood of the identity U with $\overline{xU} \subset B$ as required. □

Remark 1.38. The reals with the right order topology (Example 1.15), the open neighbourhood $(0, \infty)$ of 1 does not contain a closed neighbourhood of 1 since all non-empty closed sets in this topology contain arbitrarily large negative numbers, so 'topological' may not be weakened to 'paratopological'.

In an infinite group with the cofinite topology (Example 1.16) the only closed neighbourhood is the whole group, and so there are neighbourhoods not containing a closed neighbourhood, and so 'topological' may not be weakened to 'quasitopological'.

A topological space is said to be **Hausdorff** if for any $x \neq y$ there are disjoint open sets U and V such that $x \in U$ and $y \in V$.

Remark 1.39. A topological space has unique limits (for nets) if and only if it is Hausdorff, so this is a pretty uncontroversial axiom to want.

Remark 1.40. A subspace of a Hausdorff topological space is Hausdorff, so if H is a subgroup of a Hausdorff topological (resp. paratopological, quasitopological, semitopological) group G then H is a Hausdorff topological (resp. paratopological, quasitopological, semitopological) group when equipped with the subspace topology.

Corollary 1.41. *Suppose that G is a topological group. Then G is Hausdorff if and only if $\{1_G\}$ is closed (equivalently*[3] *if and only if G is Fréchet).*

Proof. First, if G is Hausdorff then for each $x \neq 1_G$ there is an open set U_x containing x and not containing 1_G. Hence $G \backslash \{1_G\} = \bigcup_{x \in G} U_x$ is open as required.

Conversely, if $\{1_G\}$ is closed then G is Fréchet and so for all $x \neq y$, $\{x\}$ is closed and $\{y\}$ is compact (since it is finite) so G is Hausdorff by Corollary 1.37. □

Example 1.42. An infinite group with the cofinite topology (Example 1.16) is a quasitopological group that is Fréchet but not Hausdorff, and Exercise I.1 gives an example of a paratopological group that is Fréchet but not Hausdorff.

Compact subsets of Hausdorff topological spaces are closed, and for non-Hausdorff topological groups the situation can be recovered by the next lemma.

Lemma 1.43. *Suppose that G is a topological group and K is a compact subset of G. Then $\overline{K}$ is compact.*

[3] By Lemma 1.23.

Proof. Suppose $\mathcal{U}$ is an open cover of $\overline{K}$ then by for each $x \in K$ there is an open neighbourhood of x in $\mathcal{U}$, call it U_x. By Corollary 1.37 applied to the compact set $\{x\}$ in the open set U_x there is an open neighbourhood of x, call it V_x, such that $\overline{V_x} \subset U_x$. The set $\{V_x : x \in K\}$ is an open cover of K and so by compactness has a finite subcover, say $K \subset V_{x_1} \cup \cdots \cup V_{x_k}$ and hence $\overline{K} \subset U_{x_1} \cup \cdots \cup U_{x_k}$. Thus $\mathcal{U}$ has a finite subcover of $\overline{K}$, and the result is proved. □

Remark 1.44. $\mathbb{R}$ with the right order topology (Example 1.15) has $\{0\}$ as a compact subset (since it is finite), but $\overline{\{0\}} = (-\infty, 0]$ which is *not* compact since the open cover $\{(a, \infty) : a \in \mathbb{R}\}$ has no finite subcover. In particular, we cannot relax the requirement that G be a topological group to paratopological group in Lemma 1.43.

Exercise I.3 gives an example to show that we cannot relax the hypothesis to quasitopological group either.

A topological space X is **locally compact** if every point has a compact neighbourhood.

Example 1.45. $\mathbb{Q}$ is a subgroup of $\mathbb{R}$ (with its usual topology) and so by Proposition 1.27 is a topological group with the subspace topology. However, while $\mathbb{R}$ is locally compact, $\mathbb{Q}$ is *not* locally compact. In particular, unlike the property of being Hausdorff (as covered in Remark 1.40) local compactness is *not* in general preserved on passing to subgroups.

Remark 1.46. We shall mostly be interested in locally compact Hausdorff topologies; there is a theorem of Ellis [Ell57, Theorem 2] which says that any locally compact Hausdorff *semi*topological group is a topological group, and in fact any locally compact paratopological group is a topological group.

Example 1.47 (Cofinite topologies on infinite groups, revisited)**.** Suppose that G is a group with the cofinite topology and $\mathcal{U}$ is an open cover of G. Then there is a non-empty set $U_0 \in \mathcal{U}$. Since U_0^c is finite we may write $U_0^c = \{x_1, \ldots, x_m\}$, and since $\mathcal{U}$ is a cover of G let $U_i \in \mathcal{U}$ have $x_i \in U_i$. Then $U_0, \ldots, U_m$ is a finite subcover of $\mathcal{U}$. It follows that G is compact, and hence if G is infinite then G is a compact, and *a fortiori* locally compact, quasitopological group (Example 1.16) that is not a topological group.

We shall think of locally compact topological groups as groups that are 'locally' not too large – every point has a neighbourhood that is compact – but it might otherwise be large, for example *any* group with the discrete topology is locally compact.

Lemma 1.48. *Suppose that G is a locally compact quasitopological group and K is a compact set. Then there is a symmetric open neighbourhood of the identity containing K and contained in a compact set.*

Proof. Since G is locally compact there is a compact neighbourhood of the identity L; let V be an open neighbourhood of of the identity contained in L. The set $\{xV : x \in K\}$ is an open cover of K and so there are $x_1, \dots, x_m \in K$ such that $K \subset x_1V \cup \dots \cup x_mV$; let $x_0 := 1_G$.

The result hinges on the fact that the finite union of open (resp. symmetric or compact) sets is open (resp. symmetric or compact). Since left multiplication and inversion are both assumed continuous, and the continuous image of a compact set is compact, x_iL and $(x_iL)^{-1}$ are both compact; by Lemma 1.22 x_iV and $(x_iV)^{-1}$ are open; and $x_iV \cup (x_iV)^{-1}$ is symmetric by design. It follows that $\bigcup_{i=0}^{m} (x_iV) \cup (x_iV)^{-1}$ is a symmetric open neighbourhood of the identity containing K and contained in a compact set. The result is proved. □

Remark 1.49. As it happens (see Remark 1.46) a locally compact paratopological group is necessarily a topological group (though this is by no means immediate), and *a fortiori* a quasitopological group.

A topological space X is **σ-compact** if X is a countable union of compact sets. We think of σ-compact spaces as 'globally' not too large.

Example 1.50. Since $\mathbb{Q}$ is a countable union of finite sets, *any* topology on $\mathbb{Q}$ is σ-compact. In particular, $\mathbb{Q}$ with its subspace topology (as described in Example 1.45) is σ-compact but *not* locally compact.

Corollary 1.51. *Suppose that G is a locally compact topological group. Then there is a σ-compact, locally compact open subgroup of G.*

Proof. Apply Lemma 1.48 to get a symmetric open neighbourhood of the identity S contained in a compact set L. Then $\langle S\rangle$ is a subgroup of G (Remark 1.11) which is locally compact and open by Corollary 1.25. It is contained in $\bigcup_{n\in\mathbb{N}_0} L^n$, and the latter is a countable union of compact (by Lemma 1.29) sets. The result is proved. □

The topological group of isometries of a metric space

A map $f : X \to Y$ is an **isometry** if X and Y are metric spaces with metrics d_X and d_Y respectively and $d_Y(f(x), f(y)) = d_X(x, y)$ for all $x, y \in X$.

Remark 1.52. Isometries are necessarily injective, but in general need not be surjective. Surjective isometries are exactly the isometries with an isometric inverse and are sometimes called global isometries though we shall not use this terminology.

For a metric space X we write $\mathrm{Iso}(X)$ for the set of surjective isometries $X \to X$.

Remark 1.53. Suppose that X is a topological space, Y is a set and F is a set of functions $Y \to X$. We say that F has the topology of **pointwise convergence** if it has the subspace topology it receives when considered as a subset of the set X^Y – the set of all functions $Y \to X$ – with the product topology. Equivalently this is the weakest topology on X^Y such that the maps $X^Y \to X; f \mapsto f(y)$ are continuous for all $y \in Y$.

If $\mathcal{B}$ is a base for the topology on X then a base for the topology of pointwise convergence on F is given by the sets

$$\{f \in F : f(y_i) \in U_i \text{ for all } 1 \leqslant i \leqslant n\} \text{ where } n \in \mathbb{N}_0, y_1, \ldots, y_n \in Y, \text{ and } B_1, \ldots, B_n \in \mathcal{B},$$

and the reason for the name of the topology is that $f_n \to f$ if and only if $f_n(y) \to f(y)$ in X for all $y \in Y$.

Proposition 1.54. *Suppose that X is a metric space. Then* $\mathrm{Iso}(X)$ *is a group under composition and a topological group when endowed with the topology of pointwise convergence.*

Proof. $\mathrm{Iso}(X)$ is a subset of the group of bijections $X \to X$, and by the subgroup test is a group under composition. Write d for the metric on X so that for $f_0 \in X^X$, $\epsilon > 0$ and $x_1, \ldots, x_n \in X$ the sets

$$U(f_0; \epsilon, x_1, \ldots, x_n) := \{f \in X^X : d(f(x_i), f_0(x_i)) < \epsilon \text{ for all } 1 \leqslant i \leqslant n\}$$

form a base for the topology of pointwise convergence on $\mathrm{Iso}(X)$.

For $f, g, f_0, g_0 \in X^X$ and $x \in X$ we have

$$\begin{aligned} d(g \circ f(x), g_0 \circ f_0(x)) &\leqslant d(g \circ f(x), g \circ f_0(x)) + d(g \circ f_0(x), g_0 \circ f_0(x)) \\ &= d(f(x), f_0(x)) + d(g(f_0(x)), g_0(f_0(x))), \end{aligned}$$

and hence

$$U(g_0; \epsilon/2, f_0(x_1), \ldots, f_0(x_n)) \circ U(f_0; \epsilon/2, x_1, \ldots, x_n) \subset U(g_0 \circ f_0; \epsilon, x_1, \ldots, x_n).$$

It follows that multiplication is jointly continuous. Furthermore,

$$\begin{aligned} d(g^{-1}(x), g_0^{-1}(x)) &= d(g^{-1}(g_0(g_0^{-1}(x))), g_0^{-1}(x)) \\ &= d(g(g^{-1}(g_0(g_0^{-1}(x)))), g(g_0^{-1}(x))) = d(g_0(g_0^{-1}(x)), g(g_0^{-1}(x))), \end{aligned}$$

and so $U(g_0^{-1}; \epsilon, x_1, \ldots, x_n)^{-1} = U(g_0; \epsilon, g_0^{-1}(x_1), \ldots, g_0^{-1}(x_n))$ and inversion is continuous. □

Example 1.55 (Groups of unitary maps with the strong operator topology)**.** Given an inner product space V, it is in particular a normed space with norm $\|v\| := \langle v, v\rangle^{1/2}$, and hence a metric space with metric $d(x, y) := \|x - y\|$. We write $\mathrm{U}(V)$ for the set of unitary maps from V to itself, that is the set of surjective maps $\phi : V \to V$ with $\langle \phi(v), \phi(w)\rangle = \langle v, w\rangle$ for all $v, w \in V$. $\mathrm{U}(V) \leqslant \mathrm{Iso}(V)$ where the second V is the set V with the metric d, and consider $\mathrm{U}(V)$ a topological group with topology inherited from $\mathrm{Iso}(V)$.

Remark 1.56. The space $B(V)$ of bounded linear maps $V \to V$ contains $\mathrm{U}(V)$, and the topology of pointwise convergence on $B(V)$ is called the **strong operator topology**. ⚠ Composition of maps in $B(V)$ is *not* jointly continuous in the strong operator topology, despite the fact that it *is* when restricted to $\mathrm{U}(V)$.

2 The structure-preserving maps

The structure-preserving maps that are of primary interest to us are continuous group homomorphisms.

Example 2.1. The map $\theta : \mathbb{R} \to S^1; x \mapsto \exp(2\pi i x)$ is a (surjective) continuous homomorphism.

Example 2.2. Suppose that G is a group and $\theta : G \to G$ is the identity map. If the domain is endowed with the discrete topology then θ is a continuous homomorphism whatever the topology on the codomain, and if the codomain is endowed with the indiscrete topology then similarly.

This example may seem trivial but leads to a number of counter-examples.

Example 2.3. Suppose that $\theta : \mathbb{Q} \to \mathbb{Q}$ is the identity map, with the domain discrete and the codomain the usual subspace topology inherited from $\mathbb{R}$ (as in Example 1.45). Then the domain is locally compact but the codomain is not, so local compactness is not preserved by surjective continuous group homomorphisms. (This may be compare with Corollary 2.14.)

Example 2.4. Suppose that $\theta : \mathbb{R} \to \mathbb{R}$ is the identity map, with the domain the usual topology on $\mathbb{R}$ and the codomain the indiscrete topology. Then the domain is Hausdorff and the codomain is not, so being Hausdorff is not preserved by surjective continuous group homomorphisms.

By way of contrast, surjective continuous maps take compact sets to compact sets so there is no analogous example with 'compact' in place of 'Hausdorff'.

The group structure makes checking continuity and openness a little easier:

Lemma 2.5. *Suppose that G and H are semitopological groups and $B = (B_i)_{i \in I}$ is a neighbourhood base*[4] *of the identity in H. Then a homomorphism $\theta : G \to H$ is continuous if (and only if) $\theta^{-1}(B_i)$ is a neighbourhood of the identity for all $i \in I$; and a homomorphism $\theta : H \to G$ is open if (and only if) $\theta(B_i)$ is a neighbourhood of the identity for all $i \in I$.*

[4] A **neighbourhood base** of a point x in a topological space X is a family $B = (B_i)_{i \in I}$ of neighbourhoods of x such that if U is an open set containing x then there is some $i \in I$ such that $B_i \subset U$.

Proof. Suppose that $U \subset H$ is open and $\theta(y) \in U$. By Lemma 1.22 there is an open neighbourhood of the identity V_y such that $\theta(y)V_y \subset U$. Since B is a neighbourhood base of the identity there is $i \in I$ such that $B_i \subset V_y$ and hence $\theta^{-1}(B_i) \subset \theta^{-1}(V_y)$ so $y\theta^{-1}(B_i) \subset \theta^{-1}(U)$ (using that θ is a homomorphism) and hence $\theta^{-1}(U)$ contains a neighbourhood of y *i.e.* $\theta^{-1}(U)$ is open. In the other direction, since B_i is a neighbourhood of the identity it contains an open neighbourhood of the identity which has an open set as a preimage and the identity in this preimage (since homomorphisms map the identity to the identity), whence it is an open neighbourhood of the identity and $\theta^{-1}(B_i)$ is a neighbourhood of the identity.

Now suppose that $U \subset H$ is open and $x \in \theta(U)$ so that there is some $y \in U$ such that $x = \theta(y)$. Since U is open, by Lemma 1.22 there is an open neighbourhood of the identity V_y such that $yV_y \subset U$. Since B is a neighbourhood base of the identity there is $i \in I$ such that $B_i \subset V_y$ and hence $x\theta(B_i) = \theta(yB_i) \subset \theta(U)$ (using that θ is a homomorphism). But $x\theta(B_i)$ is open by hypothesis, so $\theta(U)$ is open as required. In the other direction since B_i is a neighbourhood of the identity it contains an open set containing the identity which has an open image containing the identity (since homomorphisms map the identity to the identity), and hence the image of B_i is a neighbourhood of the identity. □

A map $\theta : G \to H$ is a **homeomorphic isomorphism** if it is both an isomorphism of the groups and a homeomorphism of the topological spaces.

Example 2.6 (Opposite groups)**.** Suppose that G is a topologized group with continuous group inversion. Write G^{op} for the **opposite group**, that is the group and topological space with the same base set, topology, identity, and inversion as G, but with multiplication $(x, y) \mapsto yx$. Then inversion is a homeomorphic isomorphism $G \to G^{\text{op}}$.

Since the map $G^2 \to G^2; (x, y) \mapsto (y, x)$ is continuous, G^{op} is a quasitopological (resp. topological) group if G is quasitopological (resp. topological).

Example 2.7 (Conjugation)**.** Suppose that G is a group. The map $G \times G \to G; (a, x) \mapsto axa^{-1}$ is a left action of G on G – it is called **conjugation**. If G is a semitopological group then for fixed a this map is a homeomorphic isomorphism of G.

Example 2.8. ⚠ There are topological groups that are isomorphic as groups and homeomorphic as topological spaces but which are *not* homeomorphically isomorphic.

Let A be the group $(\mathbb{Z}/4\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$, N be a subgroup of A isomorphic to $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$, and K a subgroup isomorphic to $\mathbb{Z}/4\mathbb{Z}$. Since A is Abelian, N (resp. K) is normal in A, and the topology $\{\varnothing, A, N, A\setminus N\}$ (resp. $\{\varnothing, A, K, A\setminus K\}$) makes A into a topological group which we denote G (resp. H).

G and H are isomorphic as groups by the identity map. Since A/N and A/K are partitions of A into sets of the same size, there is a bijection $G \to H$ that maps each set in A/N to a set in A/K. Such a map is a homeomorphism.

On the other hand, if there were a homeomorphic isomorphism $\theta : G \to H$ then the preimage of K would be either N or $G\backslash N$, but it must be the former since the identity is mapped to the identity by a group homomorphism. Thus θ restricts to a bijective homomorphism between N and K, but these are not isomorphic as groups since the latter contains an element of order 4, while the former does not.

Some useful examples of topological groups and homomorphisms between them arise through products.

Proposition 2.9. *Suppose that $(G_i)_{i\in I}$ is a family of topologized groups. Then the direct product of the groups, $\prod_{i\in I} G_i$, with the product topology is a topologized group and the projection maps $p_j : \prod_{i\in I} G_i \to G_j; x \mapsto x_j$ for each $j \in I$ are continuous open homomorphisms. Moreover, if inversion is continuous on all of the G_is then it is continuous on the product; if multiplication is separately continuous on all the G_is then it is separately continuous on the product; and if multiplication is jointly continuous on all of the G_is then it is jointly continuous on the product.*

Proof. The first part is just combines the usual results concerning product groups and topologies. The key to the 'moreover' parts is recalling the fact that the open sets in $\prod_{i\in I} G_i$ are unions of sets of the form

$$\prod_{i\in I} U_i \text{ where } \begin{cases} U_i = G_i \text{ for all } i \in I\backslash J \\ U_i \text{ is open in } G_i \text{ for all } i \in J \end{cases} \tag{2.1}$$

where J ranges all finite subsets of I. If $\prod_{i\in I} U_i$ is as in (2.1) then $(\prod_{i\in I} U_i)^{-1} = \prod_{i\in I} U_i^{-1}$ is also open if inversion is continuous on all G_i and hence inversion is continuous. Separate and joint continuity are similar. □

Remark 2.10. We call the topologized group above the **topological direct product** of the groups $(G_i)_{i\in I}$.

Remark 2.11. Given topologized groups $G_1, \dots, G_n$ we write $G_1 \times \cdots \times G_n$ for $\prod_{i\in\{1,\dots,n\}} G_i$ as usual, so for example this gives our intended meaning to $S^1 \times S^1$ in Exercise II.5.

Quotient groups

Suppose that G is a topologized group and $H \leqslant G$. Then the **quotient topology** on G/H is the strongest topology (meaning finest topology, or the topology having the most open sets) making the quotient map $q : G \to G/H; x \mapsto xH$ continuous.

Remark 2.12. For G a topologized group and $H \leqslant G$, $U \subset G/H$ is open if and only if $\bigcup U$ is open in G.

Proposition 2.13. *Suppose that G is a semitopological group and $H \leqslant G$. Then the quotient map (into G/H with the quotient topology) is open.*

Proof. If U is open in G then UH is open by Lemma 1.22. But $\bigcup q(U) = UH$, and so $q(U)$ is open in G/H by definition. $\square$

Corollary 2.14. *Suppose that G is a compact (resp. locally compact) topologized group, and $H \leqslant G$. Then G/H with the quotient topology is compact (resp. locally compact).*

Proof. For compact G this follows since the quotient map is continuous and the continuous image of a compact set is compact. Suppose G is locally compact and write q for the quotient map. Let $xH \in G/H$. Since G is locally compact there is an open set U containing x and contained in a compact set K. Since q is open, $q(U)$ is an open set containing xH and contained in $q(K)$. The latter is compact since q is continuous and so G/H is locally compact as claimed. $\square$

Proposition 2.15. *Suppose that G is a topologized group and H is a normal subgroup of G. If group inversion on G is continuous, then it is continuous on G/H; if multiplication is separately continuous on G, then it is separately continuous on G/H; and if multiplication is jointly continuous on G then it is jointly continuous on G/H. In particular if G is a topological (resp. paratopological, quasitopological, or semitopological) group then so is G/H.*

Proof. Suppose that $U \subset G/H$ is open. First suppose inversion is continuous on G. Then

$$\bigcup U^{-1} = \bigcup \{(xH)^{-1} : xH \in U\} = \bigcup \{x^{-1}H : xH \in U\} = \left\{x^{-1} : x \in \bigcup U\right\} = \left(\bigcup U\right)^{-1}$$

and so U^{-1} is open in G/H by definition since $\bigcup U$ is open in G. Secondly, suppose multiplication on G is separately continuous. For $x \in G$,

$$\bigcup (xH)^{-1}U = \bigcup \{(x^{-1}H)(yH) : yH \in U\} = \bigcup \{x^{-1}yH : yH \in U\} = x^{-1}\bigcup U,$$

and so $(xH)^{-1}U$ is open in G/H and hence left multiplication by xH is continuous. Similarly right multiplication is continuous and we are done.

Finally suppose multiplication on G is jointly continuous. Define

$$W := \{(zH, wH) \in (G/H)^2 : (zH)(wH) \in U\} \text{ and } V := \left\{(z, w) \in G^2 : zw \in \bigcup U\right\}.$$

Suppose that $(xH, yH) \in W$. Then $xy \in (xH)(yH) \subset \bigcup U$ so $(x, y) \in V$ and since V is open there are open sets $S, T \subset G$ such that $x \in S$, $y \in T$, and $S \times T \subset V$. If $s \in S$ and $t \in T$, then $st \in \bigcup U$, and since the latter is a union of cosets of H we have $(st)H \subset \bigcup U$. Since H is normal we have $(sH)(tH) = (st)H \subset \bigcup U$, and so $SH \times TH \subset V$.

By Lemma 1.22, SH and TH are open sets, and so the sets $S' := \{sH : s \in S\}$ and $T' := \{tH : t \in T\}$ are open in G/H; $xH \in S'$ and $yH \in T'$; and $S' \times T' \subset W$. It follows that W is open, and multiplication on G/H is jointly continuous. The result is proved. $\square$

Example 2.16. The topological group $\mathbb{R}$ has a (normal) subgroup $\mathbb{Z}$ and $\mathbb{R}/\mathbb{Z}$ is a topological group – it is the reals modulo 1. Moreover, the map $\mathbb{R}/\mathbb{Z} \to S^1; x + n\mathbb{Z} \mapsto \exp(2\pi i x)$ is a homeomorphic isomorphism.

⚠The notation $\mathbb{R}/\mathbb{Z}$ is sometimes (though not in these notes) used to refer to a different space: the adjunction space in which all the integers in $\mathbb{R}$ are identified but the rest of $\mathbb{R}$ remains the same. In other language this is a countably infinite bouquet of circles all connected at the point $\mathbb{Z}$.

Example 2.17. The group $\mathbb{Q}$ is a subgroup of $\mathbb{R}$ with its usual topology, and so $\mathbb{R}/\mathbb{Q}$ is a topological group. If $U \subset \mathbb{R}/\mathbb{Q}$ is open then $\bigcup U$ is open in $\mathbb{R}$ and so if it is non-empty it contains an interval I. However, $\bigcup U$ is a union of cosets of $\mathbb{Q}$ so $\bigcup U = \bigcup U + \mathbb{Q} \supset I + \mathbb{Q} = \mathbb{R}$. It follows that $\mathbb{R}/\mathbb{Q}$ is indiscrete.

⚠Note that the quotient map $q : \mathbb{R} \to \mathbb{R}/\mathbb{Q}$ is *not* closed since *e.g.* $q(\{0\}) = \{\mathbb{Q}\}$ is not closed in $\mathbb{R}/\mathbb{Q}$. This is by way of contrast with the fact that every quotient map between topological groups is open.

Topological closure preserves algebraic structure in a useful way:

Lemma 2.18. *Suppose that G is a quasitopological group and $H \leqslant G$. Then $\overline{H}$ is a subgroup of G. If G is compact then so is $\overline{H}$; if G is locally compact then so is $\overline{H}$; and if H is normal then so is $\overline{H}$.*

Proof. Suppose that $x \in H$ and $y \in \overline{H}$. If $xy \in \overline{H}^c$, then there is an open set $U \subset \overline{H}^c$ such that $xy \in U$. The set $x^{-1}U$ is an open neighbourhood of y and so there is some $h \in H$ such that $h \in x^{-1}U$ and hence (since $x \in H$, and H is a group) $U \cap H \neq \emptyset$ which is a contradiction. We conclude that $H \subset \overline{H}y^{-1}$ for all $y \in \overline{H}$. The set $\overline{H}y^{-1}$ is closed and hence contains the closure of H and so $\overline{H}^2 \subset \overline{H}$. Since inversion is continuous we have that $\overline{H}^{-1} = \overline{H}$ and $\overline{H}$ is a group.

Closed subsets of compact sets are compact so if G is compact then so is $\overline{H}$; and if G is locally compact then G has a compact neighbourhood of the identity N and hence $N \cap \overline{H}$ is a compact neighbourhood of the identity in $\overline{H}$ and so $\overline{H}$ is locally compact.

Finally, assume that H is normal. Conjugation is continuous and hence $a^{-1}\overline{H}a$ is closed for all $a \in G$, and contains $a^{-1}Ha = H$. Hence it contains the closure of H and so applying the map $x \mapsto axa^{-1}$ we get $a\overline{H}a^{-1} \subset \overline{H}$ *i.e.* $\overline{H}$ is normal. □

Remark 2.19. $\mathbb{R}$ with the right order topology (Example 1.15) has $\{0\}$ as a subgroup, but $\overline{\{0\}} = (-\infty, 0]$ which is not a subgroup so that 'quasitopological group' may not be replaced by 'paratopological group', and hence certainly may not be relaxed to 'semitopological group', in Lemma 2.18.

Paratopological groups in which the closure of every subgroup is a subgroup have been studied in [FT14].

Corollary 2.20. *Suppose that G is a topological (resp. quasitopological) group and H is a normal subgroup of G. Then $G/\overline{H}$ is a Hausdorff (resp. Fréchet) topological (resp. quasitopological) group.*

Proof. This is immediate from Lemma 2.18, Proposition 2.15, and Corollary 1.41 (resp. Lemma 1.23) for the topological (resp. quasitopological) case. □

The open mapping theorem

Example 2.2 shows that there are continuous bijective group homomorphisms that are not homeomorphic isomorphisms. This is by contrast with the purely algebraic situation where any bijective group homomorphism is a group isomorphism (*i.e.* has an inverse that is a homomorphism), but in alignment with the topological situation where continuous bijections need not be homeomorphisms. With a few mild conditions on the topology we can recover with algebraic situation:

Theorem 2.21. *Suppose that G is a σ-compact semitopological group, H is a locally compact Hausdorff topological group, and $\pi : G \to H$ is a continuous bijective homomorphism. Then π is a homeomorphic isomorphism.*

Proof. Since the inverse of a bijective group homomorphism is a group homomorphism, it suffices to show that $\pi(C)$ is closed whenever C is closed in G. Let K_n be compact in G such that $G = \bigcup_{n\in\mathbb{N}_0} K_n$.

Claim. *There is some $n \in \mathbb{N}$ such that $\pi(K_n)$ is a neighbourhood.*

Proof. For those familiar with the Baire Category Theorem this is particularly straightforward. We shall proceed directly by what is essentially the proof of the BCT for locally compact Hausdorff spaces.

Since H is Hausdorff and the sets $\pi(K_n)$ are compact (as the continuous image of compact sets), they are closed. We construct a nested sequence of closed neighbourhoods inductively: Let U_0 be a compact (and so closed since H is Hausdorff) neighbourhood in H, and for $n \in \mathbb{N}$ let $U_n \subset \pi(K_n)^c \cap U_{n-1}$ be a closed neighbourhood.

This is possible since (by the inductive hypothesis) U_{n-1} is a neighbourhood and so contains an open neighbourhood V_{n-1}. But then $\pi(K_n)^c \cap V_{n-1}$ is open and non-empty since otherwise $\pi(K_n)$ contains a neighbourhood. It follows that $\pi(K_n)^c \cap U_{n-1}$ contains an open neighbourhood and so it contains a closed neighbourhood by Corollary 1.37.

Now by the finite intersection property of the compact space U_0, the set $\bigcap_n U_n$ is non-empty. This contradicts surjectivity of π since $G = \bigcup_{n\in\mathbb{N}_0} K_n$ and the claim is proved. □

Claim. *If $X \subset H$ is compact then $\pi^{-1}(X)$ is compact.*

Proof. By the previous claim $\pi(K_n)$ contains a neighbourhood (and hence so does $x\pi(K_n)$ by Lemma 1.22) and the set $\{x\pi(K_n) : x \in H\}$ covers X, so by compactness of X there are elements $x_1, \ldots, x_m$ such that $X \subset \bigcup_{i=1}^m x_i\pi(K_n)$ and hence $\pi^{-1}(X) \subset \bigcup_{i=1}^m \pi^{-1}(x_i)K_n$ (by injectivity of π). $\pi^{-1}(x_i)K_n$ is compact by Lemma 1.22, and since a finite union of compact sets is compact it follows that $\pi^{-1}(X)$ is contained in a compact set. Finally, X is closed so $\pi^{-1}(X)$ is closed and a closed subset of a compact set is compact as required. □

Finally, suppose that $C \subset G$ is closed, and y is a limit point of $\pi(C)$. H is locally compact so y has a compact neighbourhood X. Now $\pi^{-1}(X)$ is compact and so $\pi^{-1}(X) \cap C$ is compact. But then $X \cap \pi(C)$ is compact since π is continuous, and hence closed since H is Hausdorff. But by design $y \in \overline{X \cap \pi(C)} = X \cap \pi(C) \subset \pi(C)$. □

Remark 2.22. The Open Mapping Theorem in functional analysis is the result that if $A : X \to Y$ is a surjective continuous linear operator between Banach spaces X and Y then A is an open mapping.

Remark 2.23. As with the proof of the Baire Category Theorem our argument used the axiom of dependent choices.

3 Complex-valued functions on topological groups

For a topological space X we write $C(X)$ for the set of continuous functions $X \to \mathbb{C}$.

Remark 3.1. $C(X)$ is closed under pointwise addition and multiplication of functions and contains the constant functions, so it is a $\mathbb{C}$-algebra.

⚠ Quotients of continuous functions behave a little differently: if $f, g \in C(X)$ then the support of g is open and there is a continuous function $h : \operatorname{supp} g \to \mathbb{C}$ such that $f = gh$, but in general this need[5] *not* have a continuous extension to the whole of X.

Remark 3.2. Suppose that $f \in C(X)$. By the triangle inequality if $\Delta := \{z \in \mathbb{C} : |z| < \epsilon/2\}$ and $f(x), f(y) \in z + \Delta$ then $|f(x) - f(y)| < \epsilon$ and hence $\mathcal{U} := \{f^{-1}(z + \Delta) : z \in \mathbb{C}\}$ is an open cover of X such that if $U \in \mathcal{U}$ and $x, y \in U$ then $|f(x) - f(y)| < \epsilon$.

The next result will provide a supply of continuous functions.

Theorem 3.3. *Suppose that G is a topological group, A is a compact set and B is an open set containing A. Then there is a continuous function $g : G \to [0,1]$ such that $g(x) = 0$ on for all $x \in A$ and $g(x) = 1$ for all $x \notin B$. Similarly, there is a continuous function $f : G \to [0,1]$ such that $f(x) = 1$ for all $x \in A$ and* $\operatorname{supp} f \subset B$.

[5]Consider, for example, the functions $f(x) = x$ and $g(x) = x^2$ in $C(\mathbb{R})$. Then $h(x) = 1/x$ for all $x \in \operatorname{supp} g$ but h has no continuous extension to $\mathbb{R}$.

Proof. The proof of this theorem is really a more sophisticated version of the proof of Corollary 1.37. As in the proof there we apply Lemma 1.35 to the open cover $\{B\}$ to get a symmetric open neighbourhood of the identity V such that $AV \subset B$. We may apply Lemma 1.31 twice to get a symmetric open neighbourhood of the identity V_0 such that[6] $V_0^3 \subset V$, and continue iteratively in this manner producing symmetric open neighbourhoods V_i with $V_{i+1}^3 \subset V_i$ for all $i \in \mathbb{N}_0$. In particular, note that $V_{i+1} \subset V_i$ since all the V_is are neighbourhoods of the identity.

We shall 'divide up the space between A and B' in a way that will be indexed by **dyadic rationals**, that is rationals whose denominator is a power of 2. For $i \in \mathbb{N}_0$ we write $D_i := \{q \in [0,1] : 2^i q \in \mathbb{Z}\}$, so $D := \bigcup_{i=0}^{\infty} D_i$ is the set of dyadic rationals in $[0,1]$. Note, in particular, that $D_0 \subset D_1 \subset \ldots$ and every element of $D_{i+1} \backslash D_i$ can be written uniquely in the form $\frac{1}{2}(q+q')$ where $q < q'$ are *consecutive* elements of D_i. Furthermore, in any two consecutive elements of D_{i+1}, one of them will be an element of D_i and one of $D_{i+1} \backslash D_i$.

For each $q \in D$ we define an open set U_q such that if $q < q'$ are consecutive elements of D_i for some i then $\overline{U_q} V_i \subset U_{q'}$. We proceed inductively on $i \in \mathbb{N}_0$. First, $D_0 = \{0,1\}$; let $U_0 := AV_0$ which is open by Lemma 1.22 and $U_1 := B$ which is open by definition of B. Then by Lemma 1.24 $\overline{U_0} V_0 = \overline{AV_0} V_0 \subset AV_0 V_0^{-1} V_0 \subset AV \subset B = U_1$ as required.

Suppose U_q has been defined with the required property for all $q \in D_i$. For $q < q'$ consecutive elements of D_i we define $U_{\frac{1}{2}(q+q')} := \overline{U_q} V_{i+1}$ which is open by Lemma 1.22, and furthermore by Lemma 1.24 we have $\overline{U_{\frac{1}{2}(q+q')}} V_{i+1} \subset \overline{U_q} V_{i+1} V_{i+1}^{-1} V_{i+1} \subset \overline{U_q} V_i \subset U_{q'}$. Now, if $q < q'$ are consecutive elements of D_{i+1} then either $q \in D_i$, $q'' := q + 2^i \in D_i$ and $q' = \frac{1}{2}(q+q')$; or $q' \in D_i$, $q'' := q' - 2^{-i} \in D_i$ and $q = \frac{1}{2}(q'+q'')$. In either case, by design we have $\overline{U_q} V_{i+1} \subset U_{q'}$.

We now forget about the V_is: for each $q \in D$ we have an open set U_q such that (by nesting) whenever $q < q'$ are elements of D we have $\overline{U_q} \subset U_{q'}$. Moreover, $A \subset U_0$ and $U_1 \subset B$. Define a function $g : G \to [0,1]$ by

$$g(x) := \inf \{q \in D : x \in U_q\} \text{ if } x \in U_1 \text{ and } g(x) = 1 \text{ if } x \notin U_1.$$

First note that this is well-defined and really does map into $[0,1]$. Then, since $U_1 \subset B$ we have $g(x) = 1$ for all $x \notin B$; and since $A \subset U_0$ for all $x \in A$ we have $g(x) = 0$ for $x \in A$.

It remains to establish that g is continuous. Since all open subsets of $[0,1]$ are (possibly empty) unions of finite intersections of sets of the form $[0,\alpha)$ and $(\alpha,1]$ for $\alpha \in (0,1)$, we shall show that g is continuous by showing that preimages of sets of this form are open, and we shall do *this* by showing that every point in the preimage is contained in an open neighbourhood.

First, if $x \in g^{-1}([0,\alpha))$ then $g(x) < \alpha$ and so $x \in U_1$ and by the approximation property for infima there is some $q \in D$ such that $g(x) \leqslant q < \alpha$. But then $g(z) \leqslant q < \alpha$ for all $z \in U_q$,

[6] Since $1_G \in V_0$ we certainly have $V_0^3 \subset (V_0^2)^2$.

and so $g^{-1}([0,\alpha))$ contains the open neighbourhood U_q of x as required.

Secondly, if $x \in g^{-1}((\alpha,1])$ then since D is dense in $[0,1]$ there are $q,q' \in D$ with $\alpha < q < q' < g(x)$. Hence $x \notin U_{q'}$, but $\overline{U_q} \subset U_{q'}$ by nesting and so $x \in \overline{U_q}^c$. Moreover, if $z \in \overline{U_q}^c$ then $z \notin U_q$ and so (either $z \notin U_1$ and $g(z) = 1 > \alpha$ or) $g(z) \geqslant q > \alpha$ and $g^{-1}((\alpha,1])$ contains the open neighbourhood $\overline{U_q}^c$ of x as required.

The first part is proved. For the second put $f := 1 - g$ which is continuous and maps into $[0,1]$. By design $f(x) = 1$ for all $x \in A$ and $\operatorname{supp} f \subset B$. □

Remark 3.4. The above result goes by the name 'complete regularity of topological groups' and is a slight variant of a purely topological result called Urysohn's Lemma and the proof is very similar. In particular, our argument used the axiom of dependent choice which is often used in proofs of Urysohn's Lemma.

Remark 3.5. ⚠Theorem 3.3 does *not* assume that G is not indiscrete so that there may not be *any* non-constant continuous functions. Exercise II.8 asks for a proof of this and also examples to show how things differ for quasitopological and paratopological groups.

Compactly supported continuous functions

Given a topological space X the **support** of a (not necessarily continuous) function $f : X \to \mathbb{C}$, denoted $\operatorname{supp} f$, is the set of $x \in X$ such that $f(x) \neq 0$; f is said to be **compactly supported** if its support is contained in a compact set.

Remark 3.6. ⚠As we have defined it the support of a function that is compactly supported need not actually *be* a compact set it is simply contained in one.

We write $C_c(X)$ for the subset of functions in $C(X)$ that are compactly supported.

Remark 3.7. The set $C_c(X)$ is a subalgebra of $C(X)$ since the union of two compact sets is compact and the support of the sum of two functions is contained in the union of their supports, and the support of the product of two functions is the intersection of their supports which is certainly contained in a compact set if one is. More than this, the function

$$\|f\|_\infty := \sup\{|f(x)| : x \in X\}$$

is a norm on $C_c(X)$. It is well-defined since every continuous (complex-valued) function on a compact set is bounded, and the axioms of a norm are easily checked. As a normed space $C_c(X)$ is, itself, a topological group (recall Example 1.4).

⚠In general $\|\cdot\|_\infty$ is *not* a norm on $C(X)$ since we are not assuming the elements of $C(X)$ are bounded.

⚠In general $C_c(X)$ is *not* complete despite the fact that the uniform limit of continuous functions is continuous since this limit function may not be compactly supported.

Remark 3.8. By way of contrast with the warning in Remark 3.1, if $f, g \in C_c(X)$ and $\overline{\operatorname{supp} f} \subset \operatorname{supp} g$ then there is $h \in C_c(X)$ such that $f = gh$.

Proposition 3.9. *Suppose that G is a semitopological group and $C_c(G)$ contains a function that is not identically zero. Then G is locally compact.*

Proof. Suppose that $f \in C_c(G)$ is not identically zero. Then $\operatorname{supp} f$ is open (since f is continuous), non-empty and contained in a compact set K (since f is compactly supported). It follows that K is a compact neighbourhood of some point $x \in G$, and by Lemma 1.22 $yx^{-1}K$ is then a compact neighbourhood of y for $y \in G$ as required. □

We shall be interested in the case when the functions in $C_c(X)$ can 'tell apart' the points of X: we say that a set $A \subset C_c(X)$ **separates points** if for all $x, y \in X$ with $x \neq y$ there is $f \in A$ such that $f(x) \neq f(y)$.

Remark 3.10. In particular, if G is a semitopological group and $C_c(G)$ itself separates points then G is Hausdorff and (in view of Proposition 3.9) locally compact, and so (recall Remark 1.46) G is a topological group.

For us Theorem 3.3 will be crucial in providing a supply of compactly supported functions in locally compact topological groups.

Corollary 3.11. *Suppose that G is a locally compact topological group and $K \subset G$ is compact. Then there is a continuous compactly supported $f : G \to [0, 1]$ such that $f(x) = 1$ for all $x \in K$.*

Proof. Since G is locally compact it contains a compact neighbourhood of the identity L; let $H \subset L$ be an open neighbourhood of the identity, and $C \subset H$ a closed neighbourhood of the identity (possible by Corollary 1.37). KH is open by Lemma 1.22 and apply Theorem 3.3 to get a continuous $f : G \to [0, 1]$ with $f(x) = 1$ for all $x \in K$ and $\operatorname{supp} f \subset KH \subset KL$ which is compact by Lemma 1.29. □

Furthermore, we can produce continuous partitions of unity:

Corollary 3.12. *Suppose that G is a locally compact topological group, $F : G \to [0, 1]$ is continuous, K is a compact set containing the support of F, and $\mathcal{U}$ is an open cover of K. Then there is some $n \in \mathbb{N}$ and continuous compactly supported functions $f_1, \dots, f_n : G \to [0, 1]$ such that $F = f_1 + \cdots + f_n$; and for each $1 \leqslant i \leqslant n$ there is $U_i \in \mathcal{U}$ such that* $\operatorname{supp} f_i \subset U_i$.

Proof. Since $\mathcal{U}$ is an open cover of K, for each $x \in K$ there is an open neighbourhood of x, call it $U_x \in \mathcal{U}$, and by Corollary 1.37 there is a closed neighbourhood $V_x \subset U_x$ of x. Since each V_x is a neighbourhood and $\{V_x : x \in K\}$ is a cover of K, compactness tells us that there are elements $x_1, \ldots, x_n$ such that $K \subset V_{x_1} \cup \cdots \cup V_{x_n}$. By Lemma 1.43 $\overline{K}$ is compact and so for each i the set $V_{x_i} \cap \overline{K}$ is a closed subset of a compact set and so compact. Apply Theorem 3.3 to $V_{x_i} \cap \overline{K} \subset U_{x_i}$ to get a continuous function $g_i : G \to [0,1]$ such that $g_i(x) = 1$ for all $x \in V_{x_i} \cap \overline{K}$ and $\operatorname{supp} g_i \subset U_{x_i}$.

Since the sets $V_{x_1}, \ldots, V_{x_n}$ are closed, $\overline{K} \subset V_{x_1} \cup \cdots \cup V_{x_n}$, and so since the g_is are non-negative we have

$$\overline{\operatorname{supp} F} \subset \overline{K} \subset (V_{x_1} \cap \overline{K}) \cup \cdots \cup (V_{x_n} \cap \overline{K}) \subset \operatorname{supp}(g_1 + \cdots + g_n).$$

Thus (see Remark 3.8) there is $h \in C_c(G)$ such that $F = h(g_1 + \cdots + g_n)$ and since F maps into $[0,1]$ and $g_1(x) + \cdots + g_n(x) \geqslant 1$ on the support of F, we conclude that h maps into $[0,1]$; for $1 \leqslant i \leqslant n$ put $f_i = g_i h$.

It remains to check the properties of the f_is. First, f_i is a continuous function $G \to [0,1]$ by design of h and g_i. Secondly, $F = f_1 + \cdots + f_n$ by design. Finally, $\operatorname{supp} f_i \subset \operatorname{supp} g_i \subset U_{x_i} \in \mathcal{U}$. Moreover, since the f_is are non-negative $\operatorname{supp} f_i \subset K$ so f_i has compact support. The result is proved. □

Integrals of continuous functions

We say that a complex-valued function f from a set X is **non-negative** if $f(x) \geqslant 0$ for all $x \in X$; we say a linear functional $\int$ from a complex vector space of complex-valued functions V is **non-negative** if $\int f \geqslant 0$ whenever f is non-negative.

Our motivating example of an integral is the Riemann integral:

Example 3.13. The set R of Riemann integrable functions $\mathbb{R} \to \mathbb{C}$ has some basic properties often established in first courses on analysis *e.g.* [Gre20]. In particular, R is a complex vector space under point-wise addition and scalar multiplication of functions, and

$$\int : R \to \mathbb{C}; f \mapsto \int_{-\infty}^{\infty} f(x)dx$$

is a non-negative linear map. Furthermore, $C_c(\mathbb{R})$ is a subspace of R, and $\int$ restricted to $C_c(\mathbb{R})$ is non-trivial (meaning not identically zero).

Remark 3.14. ⚠ We are only concerned with proper integrals, and though the integral in $\int$ appears to be improper we are restricting attention to compactly supported functions so the integrals are, in fact, proper.

Remark 3.15. Non-triviality of $\int$ when *restricted* to $C_c(\mathbb{R})$ is important; see Exercise III.7 for a contrasting situation.

Given a topological space X if $f, g \in C_c(X)$ are both real-valued then we write $f \geqslant g$ if $f - g$ is non-negative, and $C_c^+(X)$ for the set of $f \in C_c(G)$ such that $f \geqslant 0$, where 0 is the constant 0 function.

Remark 3.16. The functions $\mathbb{C} \to \mathbb{R}; z \mapsto \operatorname{Re} z$, $\mathbb{C} \to \mathbb{R}; z \mapsto \operatorname{Im} z$, $\mathbb{R} \to \mathbb{R}_{\geqslant 0}; x \mapsto \max\{x, 0\}$ and $\mathbb{R} \to \mathbb{R}_{\geqslant 0}; x \mapsto \max\{-x, 0\}$ are continuous and so any $f \in C_c(X)$ can be written as $f = f_1 - f_2 + if_3 - if_4$ for $f_1, f_2, f_3, f_4 \in C_c^+(X)$, and this decomposition is unique. We shall frequently have call to understand elements of $C_c(X)$ through this linear combination of elements of $C_c^+(X)$.

Remark 3.17. If $f, g \in C_c(X)$ are real-valued with $f \geqslant g$ and $\int$ is a non-negative linear functional $C_c(X) \to \mathbb{C}$ then $\int f \geqslant \int g$; and if $f \in C_c(G)$ then $\left|\int f\right| \leqslant \int |f|$.

Remark 3.18. The decomposition in Remark 3.16 can be used to show that if $\int$ is a non-negative linear functional then $\overline{\int f} = \int \overline{f}$ for all $f \in C_c(X)$.

Remark 3.19. We think of non-negative linear functionals as integrals and in fact the Riesz-Markov-Kakutani Representation Theorem actually tells us that every non-negative linear map $C_c(X) \to \mathbb{C}$ arises as an integral against a suitably well-behaved measure on X.

Given $F : X \times Y \to \mathbb{C}$ and $x \in X$ we write $\int_y F(x, y)$ for the functional $\int : C_c(Y) \to \mathbb{C}$ applied to the function $Y \to \mathbb{C}; y \mapsto F(x, y)$ (assuming this function is continuous and compactly supported), and similarly for $y \in Y$ and $\int_x F(x, y)$. It will be crucial for us that the order of integration can be interchanged and this is what the next result concerns:

Theorem 3.20 (Fubini's Theorem for continuous compactly supported functions)**.** *Suppose that G is a locally compact topological groups, $\int$ and $\int'$ are non-negative linear functionals $C_c(G) \to \mathbb{C}$, and $F \in C_c(G \times G)$. Then the map $x \mapsto \int'_y F(x, y)$ is continuous and compactly supported, so that $\int_x \int'_y F(x, y)$ exists. Similarly $y \mapsto \int_x F(x, y)$ is continuous and compactly supported, so that $\int'_y \int_x F(x, y)$ exists and moreover*

$$\int_x \int'_y F(x, y) = \int'_y \int_x F(x, y).$$

Proof. In view of the decomposition in Remark 3.16 and linearity of $\int$ and $\int'$ it is enough to establish the result for F non-negative.

Since $F \in C_c^+(G \times G)$ has support contained in a compact set K, and since the coordinate projection maps $G \times G \to G$ are continuous (and the union of two compact sets is compact) there is a compact set L such that $K \subset L \times L$. It follows that the maps $x \mapsto F(x, y)$ for $y \in G$ and $y \mapsto F(x, y)$ for $x \in G$ are continuous and have support in the compact set L.

We also need an auxiliary 'dominating function' which is a compactly supported continuous function on whose support all of the 'action' happens. For those familiar with the theory of integration, the Dominated Convergence Theorem may come to mind. Concretely, by Corollary 3.11 there is a continuous function $f : G \to [0, 1]$ with $f(x) = 1$ for all $x \in L$ supported in a compact set M.

For $\epsilon > 0$ (by Remark 3.2) let $\mathcal{U}$ be an open cover of $G \times H$ such that $|F(x, y) - F(x', y')| < \epsilon$ for all $(x, y), (x', y') \in U \in \mathcal{U}$. $M \times M$ is compact and so by Lemma 1.35 there is a symmetric open neighbourhood of the identity U in G such that $\mathcal{U}' := \{xU \times yU : x, y \in M\}$ is a refinement of $\mathcal{U}$ (as a cover of $M \times M$ *not* of $G \times G$). First, the support of $\int_y' F(x, y)$ is contained in the (compact) set L and if $x' \in xU$ then by design and non-negativity of $\int'$ we have

$$\int_y' F(x', y) = \int_y' F(x', y) f(y) \leqslant \int_y' (F(x, y) + \epsilon) f(y) = \int_y' F(x, y) + \epsilon \int' f.$$

Since U is symmetric we have $x \in x'U$ and similarly $\int_y' F(x, y) \leqslant \int_y' F(x', y) + \epsilon \int' f$ and hence $|\int_y' F(x', y) - \int_y' F(x, y)| \leqslant \epsilon \int' f$. Since ϵ is arbitrary (and $\int' f$ does not depend on ϵ) it follows that $x \mapsto \int_y' F(x, y)$ is continuous (and compactly supported) and similarly for $y \mapsto \int_x F(x, y)$.

By Corollary 3.12 applied to f supported on the compact set M with the open cover $\{xU : x \in M\}$, there are continuous compactly supported $f_1, \dots, f_n : G \to [0, 1]$ such that $f_1 + \cdots + f_n = f$ and $\operatorname{supp} f_i \subset x_i U$ for some $x_i \in M$. Now, $F(x, y) = F(x, y) f(x) f(y)$ and $f = f_1 + \cdots + f_n$, so

$$F(x, y) = \sum_{i=1}^{n} \sum_{j=1}^{n} F(x, y) f_i(x) f_j(y) \text{ for all } x, y \in G.$$

By design of $\mathcal{U}'$ and $\mathcal{U}$, for $1 \leqslant i, j \leqslant n$ there is $\lambda_{i,j} \geqslant 0$ such that $|F(x, y) - \lambda_{i,j}| < \epsilon$ for all $(x, y) \in \operatorname{supp} f_i \times \operatorname{supp} f_j$. We conclude that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_{i,j} f_i(x) f_j(y) - \epsilon f(x) f(y) \leqslant F(x, y) \leqslant \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_{i,j} f_i(x) f_j(y) + \epsilon f(x) f(y).$$

Since $\int$ and $\int'$ are non-negative linear functionals, we conclude that

$$\left| \int_x \int_y' F(x, y) - \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_{i,j} \int f_i \int' f_j \right| \leqslant \epsilon \int f \int' f$$

and

$$\left| \int_y' \int_x F(x, y) - \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_{i,j} \int f_i \int' f_j \right| \leqslant \epsilon \int f \int' f.$$

The result is proved by the triangle inequality since ϵ is arbitrary (and $\int f$ and $\int' f$ do not depend on ϵ). □

Remark 3.21. ⚠It is *not* enough to assume that $F : G \times G \to \mathbb{C}$ is such that the maps $G \to \mathbb{C}; x \mapsto \int_y' F(x, y)$ and $G \to \mathbb{C}; y \mapsto \int_x F(x, y)$ are well-defined, continuous, and compactly supported. Exercise III.4 asks for an example.

4 The Haar integral

We now turn to one of the most beautiful aspects of the theory of topological groups. This describes the way the topology and the algebra naturally conspire to produce an integral.

Given a topological group G and a function $f \in C(G)$ we write

$$\lambda_x(f)(z) := f(x^{-1}z) \text{ for all } x, z \in G.$$

Remark 4.1. $\lambda_x(f) \in C(G)$ for all $f \in C(G)$ and $x \in G$ (since left multiplication is continuous and the composition of continuous functions is continuous), and λ is a left action meaning $\lambda_{xy}(f) = \lambda_x(\lambda_y(f))$ for all $x, y \in G$ and $\lambda_{1_G}(f) = f$, and the maps λ_x are linear on the vector space $C(G)$.

⚠ Without inversion this is naturally a right action.

Remark 4.2. For a topological group G, λ restricts to an action on the space $C_c(G)$ and this action is isometric with respect to $\|\cdot\|_\infty$ *i.e.* $\|\lambda_x(f)\|_\infty = \|f\|_\infty$ for all $x \in G$.

Lemma 4.3. *Suppose that G is a topological group and $f \in C_c(G)$. Then $G \to C_c(G); x \mapsto \lambda_x(f)$ is continuous.*

Proof. Let $U \subset C_c(G)$ be open and $x \in G$ have $\lambda_x(f) \in U$. Since U is open there is $\epsilon > 0$ such that $\lambda_{x'}(f) \in U$ whenever $\|\lambda_{x'}(f) - \lambda_x(f)\|_\infty < \epsilon$.

Let K be a compact set containing the support of f. As in Remark 3.2 let $\mathcal{U}$ be an open cover of G such that $|f(x) - f(y)| < \epsilon$ for all $x, y \in U \in \mathcal{U}$. Then $\{U^{-1} : U \in \mathcal{U}\}$ is an open cover of K^{-1}. Since inversion is continuous and K is compact, K^{-1} is compact and so by Lemma 1.35 there is a symmetric open neighbourhood of the identity V such that $\{yV : y \in K^{-1}\}$ refines $\{U^{-1} : U \in \mathcal{U}\}$ (as a cover of K^{-1}), and hence $\{V^{-1}y : y \in K\}$ is a refinement of $\mathcal{U}$ (as a cover of K).

Suppose that $v \in V$ and $y \in G$ is such that $\lambda_v(f)(y) - f(y) \neq 0$. Then either $f(y) \neq 0$ so $y \in K$, but then $V^{-1}y$ is a subset of an element of $\mathcal{U}$ and so $|\lambda_v(f)(y) - f(y)| < \epsilon$; or $\lambda_v(f)(y) \neq 0$ so $v^{-1}y \in K$, but then $V(v^{-1}y) = V^{-1}(v^{-1}y)$ is a subset of an element of $\mathcal{U}$ and so again $|\lambda_v(f)(y) - f(y)| < \epsilon$. Since $\lambda_v(f) - f$ is continuous and compactly supported it attains its bounds so $\|\lambda_v(f) - f\|_\infty < \epsilon$. Finally, since λ is an action, the map λ_x is linear, and this action is isometric (Remark 4.2) we have

$$\|\lambda_{xv}(f) - \lambda_x(f)\|_\infty = \|\lambda_x(\lambda_v(f) - f)\|_\infty = \|\lambda_v(f) - f\|_\infty < \epsilon.$$

By Lemma 1.22 xV is an open neighbourhood of x and by design it is contained in the preimage of U. Since x was an arbitrary element of the preimage of U it follows this preimage is open as required. □

Given a topological group G we say that $\int : C_c(G) \to \mathbb{C}$ is a **(left) Haar integral** on G if $\int$ is a non-trivial (meaning not identically zero) non-negative linear map with

$$\int \lambda_x(f) = \int f \text{ for all } x \in G \text{ and } f \in C_c(G).$$

We sometimes call this last property **(left) translation invariance**.

Remark 4.4. Our definition of Haar integral requires $C_c(G)$ to be non-trivial and hence (*c.f.* Proposition 3.9) for G to support a Haar integral it must be locally compact. It will turn out in Theorem 4.11 that this is enough to guarantee that there is a Haar integral.

Remark 4.5. There is an analogous notion of right Haar integral which we shall not pursue here.

Example 4.6 (The Riemann Integral)**.** The map $\int$ in Example 3.13 restricted to $C_c(\mathbb{R})$ is a Haar integral. The only property not already recorded is translation invariance, and this is straightforward.

Example 4.7. If G is a discrete group then it supports a left Haar integral:

$$\int : C_c(G) \to \mathbb{C}; f \mapsto \sum_{x \in G} f(x).$$

Remark 4.8. See Exercise III.1 for a partial converse.

The integral of a non-negative continuous function that is not identically 0 is positive, and this already follows from the axioms of a Haar integral. To establish this we begin with a lemma on the comparability of functions:

Lemma 4.9. *Suppose that G is a topological group, $f, g \in C_c^+(G)$ and f is not identically zero. Then there is $n \in \mathbb{N}$, $c_1, \dots, c_n \geqslant 0$ and $y_1, \dots, y_n \in G$ such that*

$$g(x) \leqslant \sum_{i=1}^{n} c_i \lambda_{y_i}(f)(x) \text{ for all } x \in G.$$

Proof. Since $f \not\equiv 0$ there is some $x_0 \in G$ such that $f(x_0) > 0$ and hence (by Lemma 1.22) an open neighbourhood of the identity U such that $f(x_0 y) > f(x_0)/2$ for all $y \in U$. Let K be compact containing the support of g. Then $\{xU : x \in K\}$ is an open cover of K and so there are elements $x_1, \dots, x_n$ such that $x_1U, \dots, x_nU$ covers K. But then

$$g(x) \leqslant 2f(x_0)^{-1}\|g\|_\infty \sum_{i=1}^{n} f(x_0 x_i^{-1} x) = 2f(x_0)^{-1}\|g\|_\infty \sum_{i=1}^{n} \lambda_{x_i x_0^{-1}}(f)(x) \text{ for all } x \in G,$$

and the result is proved. □

Corollary 4.10. *Suppose that G is a topological group, $\int$ is a left Haar integral on G, and $f \in C_c^+(G)$ has $\int f = 0$. Then $f \equiv 0$.*

Proof. Suppose that $g \in C_c^+(G)$ so by Lemma 4.9 we have $g \leqslant \sum_{i=1}^{n} c_i \lambda_{y_i}(f)$ for $c_1, \dots, c_n \geqslant 0$ and $y_1, \dots, y_n \in G$. Then by linearity, non-negativity, and translation invariance of the Haar integral

$$\int g \leqslant \sum_{i=1}^{n} c_i \int \lambda_{y_i}(f) = \sum_{i=1}^{n} c_i \int f = 0.$$

Since $g \geqslant 0$, non-negativity of the Haar integral implies $\int g \geqslant 0$, and hence $\int g = 0$.

Now, in view of Remark 3.16 we have that $\int h = 0$ for all $h \in C_c(G)$ *i.e.* $\int$ is identically 0 contradicting the non-triviality of the Haar integral. The lemma follows. □

Existence of a Haar Integral

Our first main aim is to establish the following.

Theorem 4.11 (Existence of a Haar integral)**.** *Suppose that G is a locally compact topological group. Then there is a left Haar integral on G.*

We begin by defining a sort of approximation: for $f, \phi \in C_c^+(G)$ with ϕ not identically 0 put

$$(f;\phi) := \inf\left\{\sum_{j=1}^{n} c_j : n \in \mathbb{N}; c_1, \dots, c_n \geqslant 0; y_1, \dots, y_n \in G; \text{ and } f \leqslant \sum_{j=1}^{n} c_j \lambda_{y_j^{-1}}(\phi)\right\}. \quad (4.1)$$

We think of this as a sort of 'covering number' and begin with some basic properties:

Lemma 4.12. *Suppose that $f, g, \phi, \psi \in C_c^+(G)$ with ϕ and ψ are not identically 0. Then*

(i) $(f;\phi)$ *is well-defined;*

(ii) $(\phi;\phi) \leqslant 1$*;*

(iii) $(f;\phi) \leqslant (g;\phi)$ *whenever* $f \leqslant g$*;*

(iv) $(f + g;\phi) \leqslant (f;\phi) + (g;\phi)$*;*

(v) $(\mu f;\phi) = \mu(f;\phi)$ *for* $\mu \geqslant 0$*;*

(vi) $(\lambda_x(f);\phi) = (f;\phi)$ *for all* $x \in G$*;*

(vii) $(f;\psi) \leqslant (f;\phi)(\phi;\psi)$.

Proof. Lemma 4.9 shows that the set on the right of (4.1) is non-empty; it has 0 as a lower bound. (i) follows immediately. For (ii)[7] note that $\phi \leqslant 1.\lambda_{1_G^{-1}}(\phi)$ so that $(\phi;\phi) \leqslant 1$. (iii), (iv), (v), and (vi) are all immediate. Finally, for (vii) suppose $c_1, \dots, c_n \geqslant 0$ are such that $f \leqslant \sum_{j=1}^{n} c_j \lambda_{y_j^{-1}}(\phi)$, so that by (iii), (iv), (v), and (vi) we have $(f;\psi) \leqslant \sum_{j=1}^{n} c_j(\phi;\psi)$. The result follows on taking infima. □

To make use of $(\cdot\,;\cdot)$ we need to fix a non-zero reference function $f_0 \in C_c^+(G)$ and for $\phi \in C_c^+(G)$ not identically zero we put

$$I_\phi(f) := \frac{(f;\phi)}{(f_0;\phi)} \leqslant (f;f_0), \quad (4.2)$$

where the inequality follows from Lemma 4.12 (vii).

Many of the properties of Lemma 4.12 translate into properties of I_ϕ. In particular, we have $I_\phi(f_1 + f_2) \leqslant I_\phi(f_1) + I_\phi(f_2)$; for suitable ϕ we also have the following converse.

[7] As it happens it is easy to prove equality here but we do not need it.

Lemma 4.13. *Suppose that G is a locally compact topological group, $f_1, f_2 \in C_c^+(G)$ and $\epsilon > 0$. Then there is a symmetric open neighbourhood of the identity V such that if $\phi \in C_c^+(G)$ is not identically 0 and has support in V then $I_\phi(f_1) + I_\phi(f_2) \leqslant I_\phi(f_1 + f_2) + \epsilon$.*

Proof. Let K be a compact closed set containing the support of both f_1 and f_2 (possible since the union of two compact sets is compact and the closure of a compact set is compact by Lemma 1.43) and apply Corollary 3.11 to get $F : G \to [0,1]$ continuous, compactly supported, and with $F(x) = 1$ for all $x \in K$.

For $j \in \{1,2\}$ let g_j be continuous such that $(f_1 + f_2 + \epsilon F)g_j = f_j$ (possible in view of Remark 3.8 and use that $\overline{\operatorname{supp} f_i} \subset K \subset \operatorname{supp} F$). By Remark 3.2 (and the fact that the intersection of two open covers is an open cover) there is an open cover $\mathcal{U}$ of G such that if $x, y \in U \in \mathcal{U}$ then $|g_j(x) - g_j(y)| < \epsilon$ for $j \in \{1,2\}$. K is compact; apply Lemma 1.35 to $\mathcal{U}$ to get a symmetric open neighbourhood of the identity V such that $\{yV : y \in K\}$ refines $\mathcal{U}$ (as a cover of K).

Now suppose that $\phi \in C_c^+(G)$ is not identically 0 and has support in V, and that $c_1, \dots, c_n \geqslant 0$ and $y_1, \dots, y_n \in G$ are such that

$$f_1(x) + f_2(x) + \epsilon F(x) \leqslant \sum_{i=1}^n c_i \phi(y_i x) \text{ for all } x \in G.$$

If $\phi(y_i x) g_j(x) \neq 0$ then $x \in K$ and $y_i^{-1} \in xV$ (using $V = V^{-1}$), by xV is a subset of a set in $\mathcal{U}$ so $g_j(x) \leqslant g_j(y_i^{-1}) + \epsilon$ and hence

$$f_j(x) \leqslant \sum_{i=1}^n c_i \phi(y_i x) g_j(x) \leqslant \sum_{i=1}^n c_i (g_j(y_i^{-1}) + \epsilon)\phi(y_i x) \text{ for all } x \in G, j \in \{1,2\}.$$

By Lemma 4.12 (ii),(iii), (iv),(v) & (vi) we have

$$(f_j; \phi) \leqslant \sum_{i=1}^n c_i (g_j(y_i^{-1}) + \epsilon) \text{ for all } j \in \{1,2\},$$

but $g_1(y^{-1}) + g_2(y^{-1}) \leqslant 1$ for all $y \in G$, so

$$(f_1; \phi) + (f_2; \phi) \leqslant \sum_{i=1}^n c_i (1 + 2\epsilon).$$

Taking infima and then applying Lemma 4.12 (iv) and (v) and the inequality in (4.2) we get

$$\begin{aligned} I_\phi(f_1) + I_\phi(f_2) &\leqslant (1 + 2\epsilon) I_\phi(f_1 + f_2 + \epsilon F) \\ &\leqslant (1 + 2\epsilon)(I_\phi(f_1 + f_2) + \epsilon I_\phi(F)) \\ &\leqslant I_\phi(f_1 + f_2) + (2(f_1 + f_2; f_0) + (F; f_0) + 2\epsilon(F; f_0))\epsilon. \end{aligned}$$

The result follows since $\epsilon > 0$ was arbitrary and F, f_1, f_2 and f_0 do not depend on ϵ. □

With these lemmas we can turn to the main argument.

Proof of Theorem 4.11. By Corollary 3.11 (applied with $K = \{1_G\}$) there is $f_0 \in C_c^+(G)$ with $f_0 \not\equiv 0$. Write F for the set of functions $I : C_c^+(G) \to \mathbb{R}_{\geqslant 0}$ with $I(f) \leqslant (f; f_0)$ for all $f \in C_c^+(G)$ endowed with the product topology *i.e.* the weakest topology such that the maps $F \to [0, (f; f_0)]; I \mapsto I(f)$ are continuous for all $f \in C_c^+(G)$. Since the closed interval $[0, (f; f_0)]$ is compact, F is a product of compact spaces and so compact. Let X be the set of $I \in F$ such that

$$I(f_0) = 1 \tag{4.3}$$

$$I(\mu f) = \mu I(f) \text{ for all } \mu \geqslant 0, f \in C_c^+(G), \tag{4.4}$$

and

$$I(\lambda_x(f)) = I(f) \text{ for all } x \in G, f \in C_c^+(G). \tag{4.5}$$

The set X is closed as an intersection of the preimage of closed sets. Moreover, by Lemma 4.12 $I_\phi \in X$ for any $\phi \in C_c^+(G)$ that is not identically zero: the fact that $I(f) \in [0, (f; f_0)]$ follows from the inequality in (4.2); (4.3) by design; (4.4) by (v); and (4.5) by (vi).

This almost gives us a Haar integral (on non-negative functions) except that in general the elements of X are not additive, meaning we do not in general have $I(f+f') = I(f)+I(f')$. To get this we introduce some further sets: for $\epsilon > 0$ and $f, f' \in C_c^+(G)$ define

$$B(f, f'; \epsilon) := \{I \in X : |I(f + f') - I(f) - I(f')| \leqslant \epsilon\}.$$

As with X, the sets $B(f, f'; \epsilon)$ are closed. We shall show that any finite intersection of such sets is non-empty: For any $f_1, f_1', f_2, f_2', \ldots, f_n, f_n' \in C_c^+(G)$ and $\epsilon_1, \ldots, \epsilon_n > 0$, by Lemma 4.13 there are symmetric open neighbourhoods of the identity $V_1, \ldots, V_n$ such that if $\phi \in C_c^+(G)$ is not identically 0 and is supported in V_i then

$$|I_\phi(f_i + f_i') - I_\phi(f_i) - I_\phi(f_i')| < \epsilon_i. \tag{4.6}$$

Since G is locally compact by Lemma 1.48 there is a symmetric open neighbourhood of the identity H contained in a compact set L; set $V := H \cap \bigcap_{i=1}^n V_i$ which is also a symmetric open neighbourhood of the identity and by Theorem 3.3 there is $\phi \in C^+(G)$ that is not identically 0 with support contained in V, and hence in the compact set L which is to say it has compact support. I_ϕ enjoys (4.6) for all $1 \leqslant i \leqslant n$, and we noted before that $I_\phi \in X$, hence $I_\phi \in \bigcap_{i=1}^n B(f_i, f_i', \epsilon_i)$. We conclude that $\{B(f, f'; \epsilon) : f, f' \in C_c^+(G), \epsilon > 0\}$ is a set of closed subsets of F with the finite intersection property, but F is compact and so there is some I in all of these sets. Such an I is additive since $|I(f + f') - I(f) - I(f')| < \epsilon$ for all f, f' and $\epsilon > 0$. It remains to define $\int : C_c(G) \to \mathbb{C}$ by putting

$$\int f := I(f_1) - I(f_2) + iI(f_3) - iI(f_4) \text{ where } f = f_1 - f_2 + if_3 - if_4 \text{ for } f_1, f_2, f_3, f_4 \in C_c^+(G).$$

This decomposition of functions in $C_c(G)$ is unique (noted in Remark 3.16) and so this is well-defined. Moreover, $\int$ is linear since I is additive and enjoys (4.4); it is non-negative since I is non-negative (and $I(0) = 0$); it is translation invariant by (4.5); and it is non-trivial by (4.3). The result is proved. $\square$

Uniqueness of the Haar integral

Our second main aim is to establish the following result.

Theorem 4.14 (Uniqueness of the Haar Integral)**.** *Suppose that G is a locally compact topological group and $\int$ and $\int'$ are left Haar integrals on G. Then there is some $\lambda > 0$ such that $\int = \lambda \int'$.*

For this we introduce a little more notation: Given a topological group G and $f \in C_c(G)$ we write $\widetilde{f}(x) = \overline{f(x^{-1})}$.

Remark 4.15. $\widetilde{\cdot}$ is a conjugate-linear multiplicative involution?? on $C_c(G)$, since complex conjugation and $x \mapsto x^{-1}$ are both continuous (and continuous images of compact sets are compact).

Proof of Theorem 4.14. Suppose that $f_0, f_1 \in C_c^+(G)$ are not identically 0 and write K for a compact set containing the support of f_0 and f_1 (which exists since finite unions of compact sets are compact). By Lemma 1.48 there is a symmetric open neighbourhood of the identity, H, contained in a compact set L.

First, by Corollary 3.11 there is a continuous compactly supported function $F : G \to [0, 1]$ with $F(x) = 1$ for all $x \in KL$ (this set is compact by Lemma 1.29, and hence the corollary applies).

Now, suppose $\epsilon > 0$ and use Remark 3.2 (and the fact that intersections of open covers are open covers) to get an open cover $\mathcal{U}$ of G such that if $x, y \in U \in \mathcal{U}$ then $|f_i(x) - f_i(y)| < \epsilon$ for $i \in \{0, 1\}$. By Lemma 1.35 applied to $\mathcal{U}$ and the compact set KL there is a symmetric open neighbourhood of the identity V such that $\{xV : x \in KL\}$ is a refinement of $\mathcal{U}$ (as a cover of KL), and by Theorem 3.3 there is a continuous function $h : G \to [0, 1]$ that is not identically zero and is supported in $V \cap H$, and in particular supported in L so it has compact support.

For $x \in G$, translation invariance of $\int'$ (and Remark 3.18) tells us that

$$\int_y' h(y^{-1}x) = \int_y' \overline{\widetilde{h}(x^{-1}y)} = \overline{\int_y' \widetilde{h}(x^{-1}y)} = \overline{\int_y' \widetilde{h}(y)} = \int' \overline{\widetilde{h}}.$$

For $i \in \{0, 1\}$, the map $x \mapsto \int_y' f_i(x)h(y^{-1}x) = f_i(x) \int' \overline{\widetilde{h}}$ is continuous and is supported in K and so is compactly supported and $\int_x \int_y' f_i(x)h(y^{-1}x)$ exists and equals $\int f_i \int' \overline{\widetilde{h}}$ (by linearity of $\int$). On the other hand the map $(x,y) \mapsto f_i(x)h(y^{-1}x)$ is continuous and supported on $K \times L$ and so is compactly supported and hence by Fubini's Theorem (Theorem 3.20), $y \mapsto \int_x f_i(x)h(y^{-1}x)$ exists, and (using translation invariance of $\int$) we have

$$\int f_i \int' \overline{\widetilde{h}} = \int_x \int'_y f_i(x)h(y^{-1}x) = \int'_y \int_x f_i(x)h(y^{-1}x) = \int'_y \int_x f_i(yx)h(x).$$

Since $\{yV : y \in K\}$ refines $\mathcal{U}$ (as a cover of KL) we have $|f_i(yx) - f_i(y)| < \epsilon$ for $x \in V$ and $y \in KL$; and for $x \in H$ and $f_i(yx) \neq 0$ or $f_i(y) \neq 0$ we have $y \in KH$ whence $F(y) = 1$. It follows that

$$f_i(y)h(x) - \epsilon F(y)h(x) \leqslant f_i(yx)h(x) \leqslant f_i(y)h(x) + \epsilon F(y)h(x) \text{ for all } x, y \in G,$$

and so by non-negativity and linearity of $\int$ and $\int'$ we have

$$\int'_y \int_x f_i(y)h(x) - \int'_y \int_x \epsilon F(y)h(x) \leqslant \int'_y \int_x f_i(yx)h(x) \leqslant \int'_y \int_x f_i(y)h(x) + \int'_y \int_x \epsilon F(y)h(x).$$

It follows (using linearity of $\int$) that $|\int' f_i \int h - \int f_i \int' \overline{\widetilde{h}}| \leqslant \epsilon \int' F \int h$, and hence by the triangle inequality (and division, which is valid since $\int f_0, \int f_1 \neq 0$ by Corollary 4.10 as f_0 and f_1 are not identically zero) that

$$\left|\frac{\int' f_0}{\int f_0} - \frac{\int' f_1}{\int f_1}\right| \leqslant \left|\frac{\int' f_0}{\int f_0} - \frac{\int' \overline{\widetilde{h}}}{\int h}\right| + \left|\frac{\int' \overline{\widetilde{h}}}{\int h} - \frac{\int' f_1}{\int f_1}\right| \leqslant \epsilon \int' F \left(\frac{1}{\int f_0} + \frac{1}{\int f_1}\right).$$

Since ϵ was arbitrary (and in particular f_0, f_1, and F do not depend on it) it follows that $\int' f / \int f$ is a constant λ for all $f \in C_c^+(G)$ not identically zero. This constant must be non-zero since $\int'$ is non-trivial, and it must be positive since $\int'$ and $\int$ are non-negative. The result follows from the usual decomposition (Remark 3.16), and the fact that $\int 0, \int' 0 = 0$. □

5 The Peter-Weyl Theroem

Suppose that G is a topological group, and for an inner product space V recall the definition of $\mathrm{U}(V)$ from Example 1.55. A **finite dimensional unitary representation of** G is a continuous homomorphism $G \to \mathrm{U}(V)$ for some finite dimensional complex inner product space V.

A function $f : G \to \mathbb{C}$ is said to be a **matrix coefficient** if there is a finite dimensional unitary representation $\pi : G \to \mathrm{U}(V)$, and elements $v, w \in V$ such that $f(x) = \langle \pi(x)v, w\rangle$ for all $x \in G$.

Example 5.1. Suppose that $\pi : G \to \mathrm{U}(V)$ is a finite dimensional unitary representation of a topological group G and $e_1, \ldots, e_n$ is an orthonormal basis for V. If we write $A_{i,j} :=$

$\langle \pi(x)e_i, e_j\rangle$ and suppose that $\lambda \in \mathbb{C}^n$ is the vector for $v \in V$ written w.r.t. the basis $e_1, \dots, e_n$ (*i.e.* $\lambda_i = \langle v, e_i\rangle$), then λA – the matrix A pre-multiplied by the row vector λ – is $\pi(x)v$ written w.r.t. the basis $e_1, \dots, e_n$. This is the reason for the terminology 'matrix coefficient'.

Remark 5.2. All matrix coefficients are continuous, since continuity of $\pi : G \to \mathrm{U}(V)$ and the definition of the topology on $\mathrm{U}(V)$ means that $x \mapsto \pi(x)v$ is continuous for all $v \in V$, and the projections $v \mapsto \langle v, w\rangle$ are continuous for all $w \in V$, so the resulting composition is also continuous.

Lemma 5.3. *Suppose that G is a compact topological group. Then there is a unique left Haar integral $\int$ on G with $\int 1 = 1$ such that*

$$\langle f, g\rangle := \int f\overline{g} \text{ for all } f, g \in C(G)$$

is an inner product on $C(G)$ and for each $x \in G$, λ_x is unitary w.r.t. this inner product. Furthermore, $\|f\|_2 := \langle f, f\rangle^{1/2}$ and $\|f\|_1 := \int |f|$ define norms on $C(G)$ and

$$\|f\|_1 \leqslant \|f\|_2 \leqslant \|f\|_\infty \text{ for all } f \in C(G).$$

Proof. By Theorem 4.11 there is a left Haar integral $\int'$ on G. Since G is compact the constant function 1 is compactly supported and so by Corollary 4.10, $\int' 1 > 0$. Diving by this positive constant we get a left Haar integral $\int$ with $\int 1 = 1$. Now suppose that $\int'$ is another left Haar integral with $\int' 1 = 1$. By Theorem 4.14 $\int' = \lambda \int$ for some $\lambda > 0$, but since $\int 1 = 1 = \int' 1$ we conclude that $\lambda = 1$ and $\int = \int'$ giving the claimed uniqueness.

Linearity in the first argument and conjugate-symmetry of $\langle \cdot, \cdot\rangle$ follow from linearity of the Haar integral and Remark 3.18 respectively. $\langle f, f\rangle \geqslant 0$ for all $f \in C(G)$ since $\int$ is non-negative and $\langle \cdot, \cdot\rangle$ is then positive definite by Corollary 4.10.

The Haar integral is left-invariant so

$$\langle f, g\rangle = \int f\overline{g} = \int \lambda_x(f\overline{g}) = \int \lambda_x(f)\overline{\lambda_x(g)} \text{ for all } f, g \in C(G),$$

and the first part is proved.

For any inner product $f \mapsto \langle f, f\rangle^{1/2}$ is a norm, so $\|\cdot\|_2$ is a norm. Absolute homogeneity of $\|\cdot\|_1$ follows from the fact that the modulus of a complex number is multiplicative and $\int$ is linear, and the triangle inequality follows from, non-negativity, linearity and the triangle inequality for the modulus of a complex number. $\|f\|_1 \geqslant 0$ by non-negativity of $\int$, and finally $\|\cdot\|_1$ is positive definite by Corollary 4.10.

Since G is compact the constant functions 1 and $\|f\|_\infty^2$ are both in $C(G)$. By the Cauchy-Schwarz inequality (which holds for all inner products) we have

$$\|f\|_1 = \int |f| = \langle 1, |f|\rangle \leqslant \|1\|_2 \||f|\|_2 = \|f\|_2 \text{ for all } f \in C(G);$$

and by non-negativity of $\int$ we have

$$\|f\|_2^2 = \int |f|^2 \leqslant \int \|f\|_\infty^2 = \|f\|_\infty^2 \text{ for all } f \in C(G).$$

The result is proved. □

Remark 5.4. For the remainder of this section we write $\int$ for the unique Haar integral in Lemma 5.3, and use the notation $\langle\cdot,\cdot\rangle$, $\|\cdot\|_2$ and $\|\cdot\|_1$ as in this lemma.

Remark 5.5. Convergence in $\|\cdot\|_\infty$ is called convergence in L_∞ or **uniform convergence**; convergence in $\|\cdot\|_2$ is called convergence in L_2; and convergence in $\|\cdot\|_1$ is called convergence in L_1.

The second inequality in Lemma 5.3 tells us that uniform convergence implies convergence in L_2, and the first that convergence in L_2 implies convergence in L_1.

For $f, g \in C(G)$ we define their **convolution** to be the function

$$x \mapsto f * g(x) := \int_y f(y)g(y^{-1}x) = \langle f, \lambda_x(\widetilde{g})\rangle.$$

Lemma 5.6 (Basic properties of convolution)**.** *Suppose that G is a compact topological group. Then*

(i) $C(G) \to C(G); g \mapsto g * f$ is well-defined and linear for all $f \in C(G)$;

(ii) $h * (g * f) = (h * g) * f$ for all $f, g, h \in C(G)$;

(iii) $\lambda_x(g * f) = \lambda_x(g) * f$ for all $x \in G$, $f, g \in C(G)$;

(iv) $\langle g * f, h\rangle = \langle g, h * \widetilde{f}\rangle$ for all $f, g, h \in C(G)$ (recall $\widetilde{f}$ from just before Remark 4.15);

(v) $\|h * f\|_\infty \leqslant \min\{\|h\|_1\|f\|_\infty, \|h\|_2, \|\widetilde{f}\|_2\}$ for all $f, h \in C(G)$.

Proof. By the first part of Fubini's Theorem (Theorem 3.20) the function $g * f \in C(G)$ since $(x, y) \mapsto g(x)f(x^{-1}y)$ is continuous and compactly supported. Since $\int_x$ is linear, $g \mapsto g * f$ is well-defined and linear giving (i).

For (ii) we apply λ_y to the integrand $z \mapsto g(z)f(z^{-1}y^{-1}x)$ using that $\int_z$ is a left Haar integral; then Fubini's Theorem (Theorem 3.20) since $(z, y) \mapsto h(y)g(y^{-1}z)f(z^{-1}x)$ is continuous; and finally linearity of $\int_y$ to see that

$$\begin{aligned} h * (g * f)(x) &= \int_y h(y) \int_z g(z)f(z^{-1}y^{-1}x) \\ &= \int_y h(y) \int_z g(y^{-1}z)f(z^{-1}x) = \int_z \left(\int_y h(y)g(y^{-1}z)\right) f(z^{-1}x) = (h * g) * f(x) \end{aligned}$$

as claimed.

For (iii) note that $\lambda_t(g * f)(x) = g * f(t^{-1}x) = \langle g, \lambda_{t^{-1}x}(\widetilde{f})\rangle = \langle g, \lambda_{t^{-1}}(\lambda_x(\widetilde{f}))\rangle = \langle \lambda_t(g), \lambda_x(\widetilde{f})\rangle = \lambda_t(g) * f(x)$ since λ_t is unitary w.r.t. $\langle\cdot,\cdot\rangle$ by Lemma 5.3.

For (iv), since the function $(x, y) \mapsto g(x)f(x^{-1}y)\overline{h(y)}$ is continuous and compactly supported, by Fubini's Theorem (Theorem 3.20) and linearity of $\int_y$; and then Remark 3.18 we have

$$\begin{aligned}\langle g * f, h\rangle &= \int_y \int_x g(x)f(x^{-1}y)\overline{h(y)}\\ &= \int_x g(x) \int_y f(x^{-1}y)\overline{h(y)} = \int_x g(x) \int_y \overline{h(y)\widetilde{f}(y^{-1}x)} = \langle g, h * \widetilde{f}\rangle,\end{aligned}$$

as required.

Finally, (v) follows on the one hand since

$$|h * f(x)| \leqslant \int_y |h(y)||f(y^{-1}x)| \leqslant \int |h|\|f\|_\infty = \|h\|_1\|f\|_\infty,$$

and on the other since $|h * f(x)| = |\langle h, \lambda_x(\widetilde{f})\rangle| \leqslant \|h\|_2\|\lambda_x(\widetilde{f})\|_2 = \|h\|_2\|\widetilde{f}\|_2$. The result is proved. □

Remark 5.7. As usual, in view of the associativity in (ii) there is no ambiguity in omitting parentheses when writing expressions like $h * g * f$.

Remark 5.8. The linearity of the maps in (i) and inequality (v) mean that convolution maps convergence in L_1 to uniform convergence *c.f.* Remark 5.5.

Before beginning our main argument we need one more tool which will deal with the fact our inner product spaces are not in general complete.

Remark 5.9. A complete inner product space is called a Hilbert space and the results of this section are usually developed with respect to these. ⚠ In particular, a unitary representation is usually a continuous group homomorphism $\pi : G \to \mathrm{U}(H)$ for a complex Hilbert space H, not merely a complex inner product space. Every finite dimensional complex inner product space is complete and so a Hilbert space, and so our definition at the start of the section is not at variance with this, but in general care is warranted.

Proposition 5.10. *Suppose that G is a compact topological group G, $f \in C(G)$ and $(g_n)_{n\in\mathbb{N}}$ is a sequence of elements of $C(G)$ with $\|g_n\|_1 \leqslant 1$. Then there is a subsequence $(g_{n_i})_{i\in\mathbb{N}}$ such that $g_{n_i} * f$ converges uniformly to some element of $C(G)$ as $i \to \infty$.*

Proof. For each $j \in \mathbb{N}$, Remark 3.2 gives us an open cover $\mathcal{U}_j$ of G such that if $x, y \in U \in \mathcal{U}_j$ then $|f(x) - f(y)| < 1/j$. Since G is compact apply Lemma 1.35 to get an open neighbourhood of the identity U_j such that $\{xU_j : x \in G\}$ refines $\mathcal{U}_j$; and by compactness again there is a finite cover $\{x_{1,j}U_j, \ldots, x_{k(j),j}U_j\}$ which refines $\{xU_j : x \in G\}$. By Lemma 5.3 (v) $g_n * f(x) \in [-\|f\|_\infty, \|f\|_\infty]$. The interval $[-\|f\|_\infty, \|f\|_\infty]$ is sequentially compact, meaning every sequence has a convergent subsequence. A countable product of sequentially compact spaces is sequentially compact[8] so there is a subsequence $(n_i)_i$ such that $g_{n_i} * f(x_{k,j})$ converges, say to $g(x_{k,j})$, as $i \to \infty$ for all $1 \leqslant k \leqslant k(j)$ and $j \in \mathbb{N}$.

Suppose $\epsilon > 0$ and let $j := \lceil 3\epsilon^{-1} \rceil$. For all $1 \leqslant k \leqslant k(j)$ let M_k be such that $|g_{n_i} * f(x_{k,j}) - g(x_{k,j})| < \epsilon/6$ for all $i \geqslant M_k$; let $M := \max\{M_k : 1 \leqslant k \leqslant k(j)\}$ and suppose that $i, i' \geqslant M$.

For $x \in G$ there is some $1 \leqslant k \leqslant k(j)$ such that $x \in x_{k,j}U_j$ and hence for all $y \in G$ we have $y^{-1}x, y^{-1}x_{k,j} \in y^{-1}x_{k,j}U_j$ which is a subset of an element of $\mathcal{U}_j$, so $|f(y^{-1}x) - f(y^{-1}x_{k,j})| < 1/j$. Thus for $g \in C(G)$ with $\|g\|_1 \leqslant 1$ we have

$$\begin{aligned} |g * f(x) - g * f(x_{k,j})| &= |\langle g, \lambda_x(\widetilde{f}) - \lambda_{x_{k,j}}(\widetilde{f})\rangle| \\ &\leqslant \|g\|_1 \|\lambda_x(\widetilde{f}) - \lambda_{x_{k,j}}(\widetilde{f})\|_\infty \leqslant \sup_{y \in G} |f(y^{-1}x) - f(y^{-1}x_{j,k})| \leqslant \frac{1}{j} \leqslant \epsilon/3. \end{aligned}$$

In particular this holds for $g = g_{n_i}$ and $g = g_{n_{i'}}$, so that

$$\begin{aligned} |g_{n_i} * f(x) - g_{n_{i'}} * f(x)| \leqslant{}& |g_{n_i} * f(x) - g_{n_i} * f(x_{k,j})| + |g_{n_i} * f(x_{k,j}) - g(x_{k,j})| \\ &+ |g(x_{k,j}) - g_{n_{i'}} * f(x_{k,j})| + |g_{n_{i'}} * f(x_{k,j}) - g_{n_{i'}} * f(x)| < \epsilon. \end{aligned}$$

Since $x \in G$ was arbitrary it follows that the sequence of functions $(g_{n_i} * f)_i$ is uniformly Cauchy and so converges to a continuous function on G. The result is proved. □

We say that $V \leqslant C(G)$ is **invariant** if $\lambda_x(v) \in V$ for all $v \in V$.

Example 5.11. Suppose that $V \leqslant C(G)$ is invariant and finite dimensional. Then $\pi : G \to \mathrm{U}(V); x \mapsto (V \to V; v \mapsto \lambda_x(v))$ is a finite dimensional unitary representation.

For any $V \leqslant C(G)$ write $V^\perp$ for the set of $w \in C(G)$ such that $\langle v, w\rangle = 0$ for all $v \in V$.

Proposition 5.12. *Suppose that G is a compact group and $f \in C(G)$. Then there is an invariant space $W \leqslant C(G)$ with $\dim W \leqslant \epsilon^{-2}\|f\|_2^2$ such that if $g \in W^\perp$ then $\|g * f\|_2 \leqslant \epsilon \|g\|_2$.*

Proof. Let V be the set of vectors of the form

$$h_1 + \cdots + h_n \text{ where } n \in \mathbb{N}_0, h_i * \widetilde{f} * f = \lambda_i h_i \text{ and } \lambda_i \geqslant \epsilon^2 \text{ for all } 1 \leqslant i \leqslant n. \tag{5.1}$$

This is an invariant space by Lemma 5.6 (iii). For $v \in V$ we shall write $v = h_1 + \cdots + h_n$ to mean a decomposition as in (5.1) with the additional requirements that $h_i \not\equiv 0$ (so $\|h_i\|_2^2 \neq 0$ since h_i is continuous), and $\lambda_i \neq \lambda_j$ for $i \neq j$, which is possible since the map $T : C(G) \to C(G); h \mapsto h * \widetilde{f} * f$ is linear. (The zero vector is represented as a sum with no terms.)

[8] The proof of this is just Cantor's diagonal argument.

In fact T is positive definite and so the h_is, which are eigenvectors with corresponding eigenvalues λ_i, are perpendicular for different eigenvalues. In our language the relevant parts of this follow since if $h_i * \widetilde{f} * f = \lambda_i h_i$ and $h_j * \widetilde{f} * f = \lambda_j h_j$, then

$$\lambda_i\langle h_i, h_j\rangle = \langle \lambda_i h_i, h_j\rangle = \langle h_i * \widetilde{f} * f, h_j\rangle = \langle h_i, h_j * \widetilde{f} * f\rangle = \langle h_i, \lambda_j h_j\rangle = \overline{\lambda_j}\langle h_i, h_j\rangle.$$

Applying this identity with $j = i$ for some $h_j \neq 0$ we see that λ_i is real. Then applying it again with $\lambda_i \neq \lambda_j$ we have $\langle h_i, h_j\rangle = 0$. In particular, if $v = h_1 + \cdots + h_n$ in the way discussed after (5.1) then

$$\|v * \widetilde{f}\|_2^2 = \langle v * \widetilde{f} * f, v\rangle = \sum_{i=1}^{n} \lambda_i \|h_i\|_2^2 \geqslant \epsilon^2 \sum_{i=1}^{n} \|h_i\|_2^2 = \epsilon^2\|v\|_2^2. \tag{5.2}$$

If V contains n linearly independent vectors, then by the Gram-Schmidt process[9] there are orthonormal vectors $v_1, \ldots, v_n \in V$. For $x \in G$, by Bessel's inequality[10]

$$\sum_{i=1}^{n} |\langle v_i, \lambda_x(f)\rangle|^2 \leqslant \|\lambda_x(f)\|_2^2 = \|f\|_2^2.$$

Integrating against x and using (5.2) we have

$$n\epsilon^2 \leqslant \sum_{i=1}^{n} \int_x |v_i * \widetilde{f}(x)|^2 = \int_x \sum_{i=1}^{n} |\langle v_i, \lambda_x(f)\rangle|^2 \leqslant \int_x \|f\|_2^2 = \|f\|_2^2.$$

It follows that $\dim V \leqslant \epsilon^{-2}\|f\|_2^2$.

Write $W := \{k * \widetilde{f} : k \in V\}$, which is invariant by Lemma 5.6 (iii) and the fact V is invariant. Let $M := \sup\{\|g * f\|_2 : g \in W^\perp \text{ and } \|g\|_2 \leqslant 1\}$. We shall be done if we can show that $M^2 \leqslant \epsilon^2$.

Claim. *If $h \in V^\perp$ then $\|h * \widetilde{f}\|_2 \leqslant M\|h\|_2$.*

[9] Given $e_1, e_2, \ldots$ linearly independent, the Gram-Schmidt process in an inner product space defines

$$u_i := e_i - \sum_{k=1}^{i-1} \langle e_i, v_k\rangle v_k \text{ and } v_i := u_i/\|u_n\|.$$

It can be shown by induction that $v_1, v_2, \ldots$ is an orthonormal sequence.

[10] Bessel's inequality is the fact that if $v_1, v_2, \ldots$ is an orthonormal sequence in an inner product space then $\sum_{i=1}^n |\langle v_i, v\rangle|^2 \leqslant \|v\|^2$ for all v. To prove it note that because the v_is are orthonormal we have

$$\left\|\sum_{i=1}^{n} \langle v_i, v\rangle v_i\right\|^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} \langle v_i, v\rangle\overline{\langle v_j, v\rangle}\langle v_i, v_j\rangle = \sum_{i=1}^{n} |\langle v_i, v\rangle|^2.$$

Hence by the Cauchy-Schwarz inequality

$$\left(\sum_{i=1}^{n} |\langle v_i, v\rangle|^2\right)^2 = \left|\left\langle v, \sum_{i=1}^{n} \langle v_i, v\rangle v_i\right\rangle\right|^2 \leqslant \|v\|^2 \left\|\sum_{i=1}^{n} \langle v_i, v\rangle v_i\right\|^2 = \|v\|^2 \left(\sum_{i=1}^{n} |\langle v_i, v\rangle|^2\right).$$

Cancelling gives the inequality.

Proof. First, $h * \widetilde{f} \in W^{\perp}$: To see this, for $v \in V$ write $v = h_1 + \cdots + h_n$ to mean a decomposition as in (5.1). Then

$$\langle h * \widetilde{f}, v * \widetilde{f} \rangle = \sum_{i=1}^{n} \langle h, h_i * \widetilde{f} * f \rangle = \sum_{i=1}^{n} \lambda_i \langle h, h_i \rangle = 0.$$

Now let $k \in W^{\perp}$ have $\|k\|_2 = 1$ such that $\|h * \widetilde{f}\|_2 = \langle h * \widetilde{f}, k \rangle = \langle h, k * f \rangle \leqslant \|h\|_2 \|k * f\|_2 \leqslant M\|h\|_2$ as claimed. ☐

Let $g_n \in W^{\perp}$ have $\|g_n * f\|_2 \to M$ and $\|g_n\|_2 \leqslant 1$. By Cauchy-Schwarz we have $\|g_n\|_1 \leqslant 1$ and we may apply Proposition 5.10 to pass to a subsequence which converges uniformly. Hence by relabelling we may now additionally assume that $g_n * f \to h$ uniformly for some $h \in C(G)$. In particular, $\|g_n * f\|_2 \to \|h\|_2$ and $\langle h, g_n * f \rangle \to \|h\|_2^2$ and hence $\|h\|_2 = M$. Moreover, if $v \in V$ then $\langle g_n * f, v \rangle = \langle g_n, v * \widetilde{f} \rangle = 0$, and the former converges to $\langle h, v \rangle$, whence $h \in V^{\perp}$.

Combining this with the claim above we have

$$\begin{aligned} \|h * \widetilde{f} - M^2 g_n\|_2^2 &= \|h * \widetilde{f}\|_2^2 - 2M^2 \operatorname{Re}\langle h * \widetilde{f}, g_n \rangle + M^4 \|g_n\|_2^2 \\ &\leqslant M^2 \|h\|_2^2 - 2M^2 \operatorname{Re}\langle h, g_n * f \rangle + M^4 \to 0. \end{aligned}$$

Hence $M^2 g_n \to h * \widetilde{f}$ in $\|\cdot\|_2$, and since convergence in $\|\cdot\|_2$ is mapped to uniform convergence by convolution operations we have $M^2 g_n * f \to h * \widetilde{f} * f$. Uniqueness of limits then ensures $M^2 h = h * \widetilde{f} * f$. If $M^2 \geqslant \epsilon^2$ then $h \in V$, but then since $h \in V^{\perp}$ we see $h \equiv 0$. In that case $M = \|h\|_2 = 0$ and certainly $M^2 \leqslant \epsilon^2$ as required. The result is proved. ☐

Theorem 5.13 (The Peter-Weyl Theorem)**.** *Suppose that G is a compact topological group. Then matrix coefficients are dense in $C(G)$ with the uniform norm.*

Proof. Suppose that $f \in C(G)$ and let $\epsilon > 0$. Remark 3.2 gives us an open cover $\mathcal{U}_j$ of G such that if $x, y \in U \in \mathcal{U}_j$ then $|\widetilde{f}(x) - \widetilde{f}(y)| < \epsilon/2$. Since G is compact, by Lemma 1.35 there is an open neighbourhood of the identity U such that $\{xU : x \in G\}$ refines $\mathcal{U}$, and by Lemma 1.31 there is an open set V such that $V^2 \subset U$. By Theorem 3.3, there is $g \in C(G)$ non-negative and not identically 0 such that $\operatorname{supp} g \subset V$. By rescaling g we may assume that $\int g = 1$. The support of $g * g$ is contained in $V^2 \subset U$ and by Fubini's Theorem (Theorem 3.20) we therefore have $\int g * g = 1$. But then

$$|g * g * \overline{f}(x) - f(x)| = \left| \int_y g * g(y) \overline{f}(y^{-1}x) - \overline{f}(x) \right| = \left| \int_y g * g(y) (\widetilde{f}(x^{-1}y) - \widetilde{f}(x^{-1})) \right| \leqslant \epsilon,$$

for all $x \in G$ and so $\|\overline{f} - g * g * \overline{f}\|_{\infty} \leqslant \epsilon/2$.

Let $\delta < \epsilon \|g\|_2^{-1} \|\widetilde{f}\|_2^{-1}/2$ for reasons which will be come clear shortly. By Proposition 5.12 there is a finite dimensional invariant space $W \leqslant C(G)$ such that $\|h * g\|_2 \leqslant \delta \|h\|_2$ for all

$h \in W^\perp$. Write $\pi_W : C(G) \to C(G)$ for the map projecting onto W. Then $g - \pi_W(g) \in W^\perp$ and so $\|g * g - \pi_W(g) * g\|_2 \leqslant \delta \|g - \pi_W(g)\|_2 \leqslant \delta \|g\|_2^2$. By Lemma 5.6 (v) we have

$$\|g * g * \overline{f} - \pi_W(g) * g * \overline{f}\|_\infty \leqslant \delta \|g\|_2 \|\widetilde{f}\|_2.$$

By the triangle inequality we have $\|\overline{f} - \pi_W(g) * g * \overline{f}\|_\infty < \epsilon$. Finally, writing $k := (g * \overline{f})^\sim$ we have by definition; since λ_x is unitary; since W is invariant; since π_W is self-adjoint (meaning $\langle \pi_W v, w\rangle = \langle v, \pi_W w\rangle$ for all $v, w \in C(G)$); and again since λ_x is unitary, that

$$\begin{aligned}\pi_W(g) * g * \overline{f}(x) = \langle \pi_W(g), \lambda_x(k)\rangle &= \langle \lambda_{x^{-1}}(\pi_W(g)), k\rangle \\ &= \langle \pi_W(\lambda_{x^{-1}}(\pi_W(g))), k\rangle \\ &= \langle \lambda_{x^{-1}}(\pi_W(g)), \pi_W(k)\rangle \\ &= \langle \pi_W(g), \lambda_x(\pi_W(k))\rangle = \overline{\langle \lambda_x(\pi_W(k)), \pi_W(g)\rangle}.\end{aligned}$$

Hence $\overline{\pi_W(g) * g * \overline{f}(x)}$ is a matrix coefficient. Since $\epsilon > 0$ was arbitrary the result is proved. □

Remark 5.14. ⚠ There are other important parts to the Peter-Weyl Theorem which we have not included here.

6 The dual group

Suppose that G is a topological group. We write $\widehat{G}$ for the set of continuous homomorphisms $G \to S^1$ (where S^1 is as in Example 1.28), and call the elements of $\widehat{G}$ **characters**.

Remark 6.1. ⚠ While characters are (by definition) elements of $C(G)$, they are *not* in $C_c(G)$ unless G is compact.

We endow the set $\widehat{G}$ with the **compact-open topology**, that is the topology generated by the sets $\gamma U(K, \epsilon)$ where $\gamma \in \widehat{G}$,

$$U(K, \epsilon) := \{\lambda \in \widehat{G} : |\lambda(x) - 1| < \epsilon \text{ for all } x \in K\}$$

and $\epsilon > 0$ and K is a compact subset of G.

Proposition 6.2. *Suppose that G is a topological group. Then $\widehat{G}$ is a Hausdorff Abelian topological group with multiplication and inversion defined by*

$$(\gamma, \gamma') \mapsto (x \mapsto \gamma(x)\gamma'(x)) \text{ and } \gamma \mapsto (x \mapsto \overline{\gamma(x)}),$$

and identity the character taking the constant value 1. Moreover, $(U(K, \delta))_{K,\delta}$ as K ranges compact subsets of G and $\delta > 0$ is a neighbourhood base of the identity.

Proof. The fact that $\widehat{G}$ is an Abelian group is an easy check since S^1 is an Abelian group under multiplication and $z^{-1} = \overline{z}$ when $z \in S^1$.

Since $|\gamma(x) - 1| = |\overline{\gamma(x)} - 1|$ the inversion is certainly continuous. Now suppose that $\gamma\lambda \in \mu U(K, \epsilon)$ for some $\mu \in \widehat{G}$. Since $\gamma\lambda\overline{\mu}$ is continuous and K is compact $|\gamma\lambda\overline{\mu} - 1|$ achieves its bounds on K and hence there is some $\delta > 0$ such that $|(\gamma\lambda\overline{\mu})(x) - 1| < \epsilon - \delta$ for all $x \in K$. But then if $\gamma' \in \gamma U(K, \delta/2)$ and $\lambda' \in \lambda U(K, \delta/2)$ we have

$$\begin{aligned} |(\gamma'\lambda'\overline{\mu})(x) - 1| &\leqslant |(\gamma'\lambda'\overline{\mu})(x) - (\gamma\lambda'\overline{\mu})(x)| + |(\gamma\lambda'\overline{\mu})(x) - (\gamma\lambda\overline{\mu})(x)| + |(\gamma\lambda\overline{\mu})(x) - 1| \\ &< \delta/2 + \delta/2 + \epsilon - \delta = \epsilon. \end{aligned}$$

It follows that $\gamma'\lambda' \in \mu U(K, \epsilon)$ and so the preimage of $\gamma\lambda$ contains a neighbourhood of (γ, λ) in $\widehat{G} \times \widehat{G}$ *i.e.* multiplication is jointly continuous. Finally, the topology is Hausdorff since if $\gamma \neq \lambda$ then there is some $x \in G$ such that $\gamma(x) \neq \lambda(x)$; put $\epsilon := |\gamma(x) - \lambda(x)|/2$ and note that $\gamma U(\{x\}, \epsilon)$ and $\lambda U(\{x\}, \epsilon)$ are disjoint open sets containing γ and λ respectively. □

We call the group $\widehat{G}$ endowed with the compact-open topology the **dual group** of G, so that the above proposition tells us that if G is a topological group then its dual group is a Hausdorff Abelian topological group.

We call the identity, denoted $1_{\widehat{G}}$, the **trivial character**.

Proposition 6.3. *Suppose that G is a compact topological group. Then $\widehat{G}$ is discrete.*

Proof. Suppose that $\gamma \neq 1_{\widehat{G}}$ so there is $x \in G$ such that $\gamma(x) \neq 1$. Let $y \in G$ be such that $|\gamma(y) - 1|$ is maximal (which exists since G is compact and $x \mapsto |\gamma(x) - 1|$ is continuous) and note that by assumption this is positive. If $|\gamma(y) - 1| < 1$ then we have

$$\begin{aligned} |\gamma(y^2) - 1| = |\gamma(y)^2 - 1| &= |(2 + (\gamma(y) - 1))||\gamma(y) - 1| \\ &\geqslant (2 - |\gamma(y) - 1|)|\gamma(y) - 1| > |\gamma(y) - 1|. \end{aligned}$$

This is a contradiction, whence $\gamma \notin U(G, 1)$ and $\{1_{\widehat{G}}\}$ is open so the topology is discrete. □

Example 6.4. Suppose that G is a finite cyclic group endowed with the discrete topology. Since G is cyclic it is generated by some element x, and the map

$$\phi : G \to \widehat{G}; x^r \mapsto (G \to S^1; x^l \mapsto \exp(2\pi i r l/|G|))$$

is a well-defined homeomorphic isomorphism. To see this note that ϕ is well-defined in the sense that different representations of an element in the domain product the same image: since $x^r = x^{r'}$ implies $|G| \mid r - r'$ and hence $\exp(2\pi i r l/|G|) = \exp(2\pi i r' l/|G|)$; and ϕ is well-defined in the sense that $\phi(x^r)$ as defined is genuinely an element of $\widehat{G}$: $x^l = x^{l'}$ implies $|G| \mid l - l'$ and hence $\exp(2\pi i r l/G|) = \exp(2\pi i r l'/|G|)$ so that $\phi(x^r)$ is itself a well-defined function; it is continuous since G is discrete; and it is a homomorphism since $\exp(2\pi ir(l+l')/|G|) = \exp(2\pi irl/|G|)\exp(2\pi irl'/|G|)$.

ϕ is a homomorphism since $\exp(2\pi i(r+r')l/|G|) = \exp(2\pi irl/|G|)\exp(2\pi ir'l/|G|)$. ϕ is injective since if $\exp(2\pi irl/|G|) = 1$ for all l then $|G| \mid r$ so $x^r = 1_G$. ϕ is surjective since if $\gamma : G \to S^1$ is a homomorphism then $\gamma(x)^{|G|} = 1$ so $\gamma(x) = \exp(2\pi ir/|G|)$ for some $r \in \mathbb{Z}$, and $\gamma = \phi(x^r)$.

We conclude that $\phi : G \to \widehat{G}$ is a bijective group homomorphism and hence ϕ^{-1} is a group homomorphism. Since G is discrete ϕ is continuous. Since G is finite, G is compact and so $\widehat{G}$ is discrete by Proposition 6.3 and hence ϕ^{-1} is continuous as required.

In particular G and $\widehat{G}$ are homeomorphically isomorphic.

Remark 6.5. Example 6.4 gives a class of topological groups that are homeomorphically isomorphic to their duals. Since there are non-Abelian groups, and the dual group is always Abelian (Proposition 6.2), there are many examples where a group and its dual are not even isomorphic. Similarly, since there are non-Hausdorff topological groups and the dual group is always Hausdorff (Proposition 6.2), there are many examples where a group and its dual are not homeomorphic as topological spaces.

It will turn out that it is more natural to ask when a group and its double dual are homeomorphically isomorphic and there will be a wide class of groups where this will hold.

Example 6.6. When G is a group with the indiscrete topology the only continuous functions are constant and so $\widehat{G}$ is the trivial group with one character taking the constant value 1 (and there is only one topology on a set with one element) so that we have completely determined the topological group $\widehat{G}$.

Example 6.6 gave topological reasons for the dual group being trivial, but there can also be algebraic reasons:

Example 6.7 (Non-Abelian finite simple groups)**.** Suppose that G is a non-Abelian finite simple[11] topological group.

Suppose that $\gamma : G \to S^1$ is a homomorphism. Since G is non-Abelian there are elements $x, y \in G$ with $xy \neq yx$, but then $xyx^{-1}y^{-1} \neq 1_G$ while

$$\gamma(xyx^{-1}y^{-1}) = \gamma(x)\gamma(y)\gamma(x)^{-1}\gamma(y)^{-1} = 1$$

since S^1 *is* Abelian. We conclude that the kernel of γ is non-trivial, but all kernels are normal subgroups and since G is simple it follows that $\ker\gamma = G$ *i.e.* γ is trivial. In other words $\widehat{G} = \{1_{\widehat{G}}\}$.

[11]A **simple group** is a group whose only normal subgroups are the trivial group and the whole group *e.g.* A_n, the alternating group on n elements, when $n \geqslant 5$. (The Abelian finite simple groups are the cyclic groups of prime order and their dual groups are described in Example 6.4.)

The topology on G and $\widehat{G}$ are quite closely related: if G is compact then $\widehat{G}$ is discrete (Proposition 6.3), and the other way round we have the following:

Proposition 6.8. *Suppose that G is a discrete topological group. Then $\widehat{G}$ is compact.*

Proof. The set $\widehat{G}$ is a subset of the topological space M of functions $G \to S^1$ endowed with the product topology, which itself is compact by Tychonoff's theorem. (*c.f.* the set F considered in the proof of Theorem 4.11.). Since G is discrete the only compact sets in G are finite and hence the topology on $\widehat{G}$ is the subspace topology induced by viewing it as a subspace of M. It remains to check that $\widehat{G}$ is closed at which point it follows that it is compact. To see it is closed, note that the sets $\{f : G \to S^1 : f(xy) = f(x)f(y)\}$ are closed for each $x, y \in G$, and hence

$$\bigcap \{\{f : G \to S^1 : f(xy) = f(x)f(y)\} : x, y \in G\}$$

is closed. This is the set of all homomorphisms $G \to S^1$, but *every* homomorphism is continuous since G is discrete and hence this set equals $\widehat{G}$. □

We can make use of the Haar integral we have developed to show that if G is a locally compact topological group then the dual group is also locally compact. To do this we need a lemma.

Lemma 6.9. *Suppose that G is a locally compact topological group supporting a Haar integral $\int$, $f_0 \in C_c^+(G)$ has $\int f_0 \neq 0$, and $\kappa, \delta > 0$. Then there is an open neighbourhood of the identity $L_{\delta,\kappa}$ such that if $\left|\int f_0\gamma\right| \geqslant \kappa \int f_0$ then $|1 - \gamma(y)| < \delta$ for all $y \in L_{\delta,\kappa}$.*

Proof. By Lemma 4.3 there is an open neighbourhood of the identity $L_{\delta,\kappa}$ (which we may assume is contained in U since U is a neighbourhood and so contains an open neighbourhood of the identity) such that $\|\lambda_y(f_0) - f_0\|_\infty < \delta\kappa / \int F$ for all $y \in L_{\delta,\kappa}$. (Note $\int F > 0$ by Corollary 4.10.) For $y \in L_{\delta,\kappa}$, the support of $\lambda_y(f_0) - f_0$ is contained in UK (since $L_{\delta,\kappa} \subset U$) and so

$$\int |\lambda_y(f_0) - f_0| \leqslant \|\lambda_y(f_0) - f_0\|_\infty \int F < \delta\kappa.$$

Now, if $y \in L_{\delta,\kappa}$ then

$$\begin{aligned} |1 - \gamma(y)|\kappa \leqslant \left|(\gamma(y) - 1)\int f_0\gamma\right| &= \left|\int f_0 \lambda_{y^{-1}}(\gamma) - \int f_0\gamma\right| \\ &= \left|\int \lambda_y(f_0)\gamma - \int f_0\gamma\right| \leqslant \int |\lambda_y(f_0) - f_0| < \delta\kappa. \end{aligned}$$

Dividing by κ gives the claim. □

Theorem 6.10. *Suppose that G is a locally compact topological group. Then $\widehat{G}$ is locally compact.*

Proof. Let $\int$ be a left Haar integral on G (which exists by Theorem 4.11). Since $\int$ is non-trivial there is $f_0 \in C_c^+(G)$ such that $\int f_0 \neq 0$ and we may rescale so that $\int f_0 = 1$. Write K for a compact set containing the support of f_0 and U for a compact neighbourhood of the identity.

UK is compact by Lemma 1.29. Apply Corollary 3.11 to get a continuous compactly supported $F : G \to [0, 1]$ such that $F(x) = 1$ for all $x \in UK$. Define

$$V := \{\gamma \in \widehat{G} : |\gamma(x) - 1| \leqslant 1/4 \text{ for all } x \in K\},$$

so that V certainly contains, $U(K, 1/4)$, an open neighbourhood of the identity.

As in the proof of Proposition 6.8 we write M for the set of maps $G \to S^1$ endowed with the product topology so that M is compact. As sets $\widehat{G}$ is contained in M, but the compact-open topology on $\widehat{G}$ is *not*, in general, the same as that induced on $\widehat{G}$ as a subspace of M. Our aim is to make use of the compactness on M to show that $\widehat{G}$ is locally compact in the compact-open topology.

First we restrict to homomorphisms: write H for the set of homomorphisms $G \to S^1$, which is a closed subset of M since it is the intersection over all pairs $x, y \in G$ of the set of $f \in M$ such that $f(xy) = f(x)f(y)$. Write

$$C := \bigcap_{\delta > 0, x \in L_{\delta,3/4}} \{f \in H : |f(x) - 1| \leqslant \delta\}$$

which is also closed as an intersection of closed sets. By Lemma 2.5 as sets we have $C \subset \widehat{G}$ since the sets $\{z \in S^1 : |1 - z| \leqslant \delta\}$ form a neighbourhood base of the identity in S^1, and if $f \in C$ then $f^{-1}(\{z \in S^1 : |1 - z| \leqslant \delta\}) \supset L_{\delta,3/4}$ which is a neighbourhood of the identity in G.

If $\gamma \in V$ then $\left|1 - \int f_0\gamma\right| \leqslant \int f_0|1 - \gamma| \leqslant 1/4$, so by the triangle inequality $|\int f_0\gamma| \geqslant 3/4$ and hence the claim tells us that $\gamma \in C$. Thus (as sets) $V \subset C \subset \widehat{G}$ and so

$$V = \bigcap_{x \in K} \{f \in C : |f(x) - 1| \leqslant 1/4\},$$

which is again a closed subset of M.

Our aim is to show that V is compact in the compact-open topology on $\widehat{G}$. This follows if every cover of the form $\mathcal{U} = \{\gamma U(K_\gamma, \delta_\gamma) : \gamma \in V\}$ (where K_γ is compact and $\delta_\gamma > 0$) has a finite subcover. Write $L_\gamma := L_{\delta_\gamma/2,1/2}$ and note that by compactness of K_γ there is a finite set T_γ such that $K_\gamma \subset T_\gamma L_\gamma$. Write

$$U_\gamma := \{f \in M : |f(x) - 1| < \delta_\gamma/2 \text{ for all } x \in T_\gamma\}$$

which is an open set in M since T_γ is finite. Suppose that $\lambda \in (\gamma U_\gamma) \cap V$. Then since

$\gamma, \lambda \in V$, the triangle inequality gives

$$\left|1 - \int f_0 \overline{\gamma}\lambda\right| \leqslant \int f_0 |1 - \overline{\gamma}\lambda| = \int f_0 |1 - \overline{\gamma} + \overline{\gamma} - \overline{\gamma}\lambda|$$
$$\leqslant \int f_0 |1 - \gamma| + \int f_0 |1 - \lambda| \leqslant 1/2.$$

Hence $\left|\int f_0 \overline{\gamma}\lambda\right| \geqslant 1/2$ by the triangle inequality again. The claim gives $|1 - \overline{\gamma(y)}\lambda(y)| < \delta_\gamma/2$ for all $y \in L_\gamma$. But $\overline{\gamma}\lambda \in U_\gamma$ so we also have $|1 - \overline{\gamma(z)}\lambda(z)| < \delta_\gamma/2$ for all $z \in T_\gamma$. Thus, if $x \in K_\gamma$ then there is $z \in T_\gamma$ and $y \in L_\gamma$ such that $x = zy$ and

$$|1 - \overline{\gamma(x)}\lambda(x)| \leqslant |1 - \overline{\gamma(z)}\lambda(z)| + |\overline{\gamma(z)}\lambda(z) - \overline{\gamma(zy)}\lambda(zy)|$$
$$= |1 - \overline{\gamma(z)}\lambda(z)| + |1 - \overline{\gamma(y)}\lambda(y)| < \delta_\gamma.$$

We conclude that $\gamma U_\gamma \cap V \subset \gamma U(K_\gamma, \delta_\gamma) \cap V$. Finally $\{\gamma U_\gamma : \gamma \in V\}$ is a cover of V by sets that are open in M. M is compact and V is closed as a subset of M so V is compact as a subset of M, and hence $\{\gamma U_\gamma : \gamma \in V\}$ has a finite subcover which leads to a finite subcover of our original cover $\mathcal{U}$. The result is proved. □

Remark 6.11. The above shows that the dual of a locally compact Hausdorff Abelian topological group is a locally compact Hausdorff Abelian topological group. Pontryagin duality is a powerful strengthening of this in which a crucial part is showing that characters separate points. This can be deduced from the Peter-Weyl Theorem.

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