GEOMETRY – SHEET 2 – Vector Product. Vector Algebra.
(Exercises on lectures in Week 3)

1. Write the equations of each of the following lines in the form $\mathbf{r} \wedge \mathbf{a} = \mathbf{b}$.

(i) The line through the points $(1,1,1)$ and $(1,2,3)$.

(ii) The line with equation $(x-1)/2 = y/3 = z+1$.

(iii) The intersection of the planes $x+y+z=1$ and $x-y-z=2$.

2. Let

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \qquad M = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}.$$

Show that $\mathbf{a} \wedge \mathbf{b} = M\mathbf{b}$ and use the vector triple product to show that $M^3 = -|\mathbf{a}|^2 M$.

3. (i) Let A, B, C be three points in space with position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ from an origin O. Show that A,B and C are collinear if and only if

$$\mathbf{a} \wedge \mathbf{b} + \mathbf{b} \wedge \mathbf{c} + \mathbf{c} \wedge \mathbf{a} = \mathbf{0}.$$

(ii) Show that the equation of the plane containing three non-collinear points with position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is

$$\mathbf{r} \cdot (\mathbf{a} \wedge \mathbf{b} + \mathbf{b} \wedge \mathbf{c} + \mathbf{c} \wedge \mathbf{a}) = [\mathbf{a}, \mathbf{b}, \mathbf{c}].$$

Deduce that four points A, B, C, D with respective position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ are coplanar if and only if

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] - [\mathbf{b}, \mathbf{c}, \mathbf{d}] + [\mathbf{c}, \mathbf{d}, \mathbf{a}] - [\mathbf{d}, \mathbf{a}, \mathbf{b}] = 0.$$

4. (i) Let A be a 3×3 matrix and $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be three column vectors in $\mathbb{R}^3$. Use the determinant product rule to show that

$$[A\mathbf{u}, A\mathbf{v}, A\mathbf{w}] = \det A \times [\mathbf{u}, \mathbf{v}, \mathbf{w}].$$

(ii) Let T be the tetrahedron with vertices $(0,0,0)$, $(0,0,1)$, $(0,1,0)$ and $(1,0,0)$. Let $0 < c < 1$. Show that the triangular intersection of T with the plane $z = c$ has area $(1-c)^2/2$. Hence find the volume of T.

(iii) Deduce that the volume of the tetrahedron with vertices $\mathbf{0}, \mathbf{u}, \mathbf{v}, \mathbf{w}$ is given by $|[\mathbf{u}, \mathbf{v}, \mathbf{w}]|/6$.

5. (i) Let $\mathbf{a}$ and $\mathbf{b}$ be independent vectors in $\mathbb{R}^3$. Show that $|\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2 = |\mathbf{a} \wedge \mathbf{b}|^2 \neq 0$.

(ii) Using the fact that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{a} \wedge \mathbf{b}$ form a basis of $\mathbb{R}^3$, or otherwise, show that the planes

$$\mathbf{r} \cdot \mathbf{a} = \alpha, \qquad \mathbf{r} \cdot \mathbf{b} = \beta,$$

intersect in a line parallel to $\mathbf{a} \wedge \mathbf{b}$.

(iii) Under what conditions do the equations in (ii) and the equation $\mathbf{r} \cdot \mathbf{c} = \gamma$ (where $\mathbf{c} \neq \mathbf{0}$) have a unique common solution?

6. (*Optional*) Two non-parallel lines l_1 and l_2 in three-dimensional space have respective equations $\mathbf{r} \wedge \mathbf{a}_1 = \mathbf{b}_1$ and $\mathbf{r} \wedge \mathbf{a}_2 = \mathbf{b}_2$.

For $i = 1, 2$ let Π_i denote the plane of the form $\mathbf{r} \cdot (\mathbf{a}_1 \wedge \mathbf{a}_2) = k_i$ which contains l_i. Show that $k_1 = \mathbf{b}_1 \cdot \mathbf{a}_2$ and find k_2. Hence show that the least distance between the lines equals

$$\frac{|\mathbf{a}_1 \cdot \mathbf{b}_2 + \mathbf{a}_2 \cdot \mathbf{b}_1|}{|\mathbf{a}_1 \wedge \mathbf{a}_2|}.$$