GEOMETRY – SHEET 3 – Conics. 2×2 **Orthogonal Matrices.**
(Exercises on lectures in Week 4)

1. Find the equations in Cartesian co-ordinates of:

(i) the ellipse with foci at $(\pm 2, 0)$ which passes through $(0, 1)$;

(ii) the hyperbola with asymptotes $y = \pm 2x$ and directrices $x = \pm 1$;

(iii) the ellipse consisting of all points P such that $|AP| + |BP| = 10$, where A is $(3, 0)$ and B is $(-3, 0)$;

(iv) the parabola with directrix $x + y = 1$ and focus $(-1, -1)$.

[Hint: the point (x_0, y_0) is at a distance $|ax_0 + by_0 + c| / \sqrt{a^2 + b^2}$ from the line $ax + by + c = 0$.]

2. Consider the parabola C with equation $y^2 = 4ax$ and let point $P = (at^2, 2at)$. What is the gradient of C at P in terms of t?

The focus F is at $(a, 0)$. Let l_1 denote the line connecting P and F, and θ_1 denote the angle between l_1 and the tangent to C at P. Show that

$$\cos\theta_1 = \frac{t}{\sqrt{t^2 + 1}}.$$

Let l_2 denote the horizontal line through P and θ_2 denote the angle between l_2 and the tangent to C at P. Show that $\theta_1 = \theta_2$. (Consequently any light beam emitted from the focus F will reflect horizontally after contact with a parabolic mirror.)

3. (i) The conic C is formed by intersecting the double cone $x^2 + y^2 = z^2$ with the plane $x + y + z = 1$. Show that the point with position vector

$$\mathbf{r}(t) = (1 + (\sec t - \tan t)/\sqrt{2},\ 1 + (\sec t + \tan t)/\sqrt{2},\ -1 - \sqrt{2}\sec t)$$

lies on C.

(ii) Show that the vectors $\mathbf{e}_1 = (1/\sqrt{6}, 1/\sqrt{6}, -\sqrt{2/3})$ and $\mathbf{e}_2 = (-1/\sqrt{2}, 1/\sqrt{2}, 0)$ are of unit length, are perpendicular to one another and are parallel to the plane $x + y + z = 1$. Show further that

$$\mathbf{r}(t) = (1, 1, -1) + (a\sec t)\mathbf{e}_1 + (b\tan t)\mathbf{e}_2$$

where a and b are positive numbers to be determined.

(iii) Show that C has eccentricity $2/\sqrt{3}$, has foci $(1, 1, -1) \pm 2\mathbf{e}_1$ and that the directrices, in parametric form, are $(1, 1, -1) \pm 3\mathbf{e}_1/2 + \lambda\mathbf{e}_2$.

4. By rotating the xy-axes appropriately, and subsequently completing the squares, show that the curve

$$6x^2 + 4xy + 9y^2 - 12x - 4y - 4 = 0,$$

is an ellipse and find its area.

5. (i) Describe (without proof) all the 2×2 orthogonal real matrices.

(ii) Given that every isometry of $\mathbb{R}^2$ can be written in the form $\mathbf{x} \mapsto A\mathbf{x} + \mathbf{b}$, where A is orthogonal and $\mathbf{b} \in \mathbb{R}^2$, show that every isometry of $\mathbb{C}$ can be written in the form

$$z \mapsto az + b \qquad \text{or} \qquad z \mapsto a\bar{z} + b \qquad (*)$$

where $a, b \in \mathbb{C}$ and $|a| = 1$.

(iii) Write the reflection in the line $x + y = 1$ in one of the above forms given in $(*)$.

6. (*Optional*) Let $\mathbf{v}$ and $\mathbf{w}$ be independent vectors in $\mathbb{R}^2$. Show that

$$\mathbf{r}(t) = \mathbf{v}\cos t + \mathbf{w}\sin t,$$

where $0 \leqslant t < 2\pi$, is a parametrization of an ellipse.