GEOMETRY – SHEET 4 – Orthogonal Change of Coordinates. 3×3 Orthogonal Matrices.
(Exercises on lectures in Week 5)

1. (i) Let

$$A = \begin{pmatrix} 1 & a & b \\ c & d & -1 \\ e & \frac{1}{2} & f \end{pmatrix}.$$

Are there constants a, b, c, d, e, f such that A is orthogonal?

(ii) If an orthogonal matrix represents a reflection, show that it is symmetric. Is the converse true?

(iii) Let A be an $n \times n$ matrix for which there exists an orthogonal matrix P such that $P^T AP$ is diagonal. Show that A is symmetric.

2. Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be an orthonormal basis in $\mathbb{R}^3$ which is right-handed (so that $\mathbf{e}_1 \wedge \mathbf{e}_2 = \mathbf{e}_3$, $\mathbf{e}_2 \wedge \mathbf{e}_3 = \mathbf{e}_1$, $\mathbf{e}_3 \wedge \mathbf{e}_1 = \mathbf{e}_2$). Say that

$$X\mathbf{e}_1 + Y\mathbf{e}_2 + Z\mathbf{e}_3 = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \qquad \text{and} \qquad \tilde{X}\mathbf{e}_1 + \tilde{Y}\mathbf{e}_2 + \tilde{Z}\mathbf{e}_3 = \tilde{x}\mathbf{i} + \tilde{y}\mathbf{j} + \tilde{z}\mathbf{k}.$$

Show that

$$X\tilde{X} + Y\tilde{Y} + Z\tilde{Z} = x\tilde{x} + y\tilde{y} + z\tilde{z}$$

and

$$(Y\tilde{Z} - Z\tilde{Y})\mathbf{e}_1 + (Z\tilde{X} - X\tilde{Z})\mathbf{e}_2 + (X\tilde{Y} - Y\tilde{X})\mathbf{e}_3 = (y\tilde{z} - z\tilde{y})\mathbf{i} + (z\tilde{x} - x\tilde{z})\mathbf{j} + (x\tilde{y} - y\tilde{x})\mathbf{k}.$$

What is the significance of these identities?

3. Let $\mathbf{n}$ be a unit vector in $\mathbb{R}^3$. The *stretch* S_k with invariant plane $\mathbf{r} \cdot \mathbf{n} = c$ and stretch factor k is defined by

$$S_k(\mathbf{v}) = \mathbf{v} + (k-1)\,(\mathbf{v} \cdot \mathbf{n} - c)\,\mathbf{n}.$$

(i) Describe the maps S_1, S_0 and S_{-1}.

(ii) Determine the matrix for S_k when the invariant plane is $x + y + z = 0$.

(iii) Find an orthonormal basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ of $\mathbb{R}^3$ such that $\mathbf{e}_1, \mathbf{e}_2$ are parallel to the plane in (ii). Describe the map S_k in terms of the co-ordinates associated with $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

4. Consider the orthogonal matrices

$$A = \frac{1}{3}\begin{pmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{pmatrix}; \qquad B = \frac{1}{2}\begin{pmatrix} \sqrt{2} & \sqrt{2} & 0 \\ 1 & -1 & \sqrt{2} \\ 1 & -1 & -\sqrt{2} \end{pmatrix}.$$

Is either a rotation? – in which case find the axis and angle of rotation. Is either a reflection? – in which case find the plane of reflection.

5. With $0 \leqslant \theta < 2\pi$, let

$$A_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}, \qquad \text{and} \qquad B = \frac{1}{25}\begin{pmatrix} 15 & 0 & 20 \\ -16 & 15 & 12 \\ 12 & 20 & -9 \end{pmatrix}.$$

(i) Show that B is orthogonal and that $\det B = -1$. Show that B does not represent a reflection.

(ii) Find a value of θ such that $A_\theta B$ represents a reflection. For this value of θ, find the plane of reflection of $A_\theta B$.

6. (*Optional*) A *quaternion* is a number of the form $Q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ where a, b, c, d are real and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ satisfy

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1.$$

Quaternions then add and multiply as one would expect (associatively and distributively), with real scalars commuting.

(i) Show that $\mathbf{ij} = \mathbf{k}$ and that $\mathbf{ij} = -\mathbf{ji}$. (Hence quaternion multiplication is not commutative.)

(ii) For such Q we write $\overline{Q} = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}$ and set $|Q|^2 = Q\overline{Q}$. Show that $|Q|^2$ is real and non-negative.

(iii) A quaternion P is said to be *pure* if it is a combination of $\mathbf{i}, \mathbf{j}, \mathbf{k}$. Show that for pure quaternions P_1, P_2 we have $P_1P_2 = -P_1 \cdot P_2 + P_1 \wedge P_2$.

(iv) Let

$$Q = \sqrt{3}/2 + (\mathbf{i} + \mathbf{j} + \mathbf{k})/(2\sqrt{3}).$$

Show that $|Q| = 1$. If we identify the pure quaternions with $\mathbb{R}^3$ and consider the map $T(P) = QP\overline{Q}$, then show that T represents a rotation by $\pi/3$ about the line $x = y = z$.