GEOMETRY – SHEET 5 – Isometries. Rotating Frames
(Exercises on lectures in Week 6)

1. (i) Let $\mathbf{r} = r\mathbf{e}_r$ in $\mathbb{R}^2$. If r remains constant, show that $\dot{\mathbf{r}} = \dot{\theta}\mathbf{k} \wedge \mathbf{r}$.

(ii) A particle travels on the spiral $r = e^{\theta}$. Show that the angle the particle's velocity makes with the radial vector $\mathbf{r}$ is constant.

2. (i) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be the map given by

$$T(\mathbf{x}) = A\mathbf{x} + \mathbf{b} \quad \text{where} \quad A = \begin{pmatrix} 1/3 & 0 \\ 2/3 & 1/\sqrt{2} \\ 2/3 & -1/\sqrt{2} \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Show that $A^T A = I_2$ and deduce that T is an isometry. What is the image of T?

(ii) Show that there is no isometry from $\mathbb{R}^3$ to $\mathbb{R}^2$.

3. Let

$$B = \frac{1}{2}\begin{pmatrix} \sqrt{2} & \sqrt{2} & 0 \\ 1 & -1 & \sqrt{2} \\ 1 & -1 & -\sqrt{2} \end{pmatrix}, \qquad R(\mathbf{i},\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}, \qquad R(\mathbf{j},\theta) = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix},$$

where $\theta \in \mathbb{R}$. Find α, β, γ in the ranges $-\pi < \alpha \leqslant \pi$, $0 \leqslant \beta \leqslant \pi$ and $-\pi < \gamma \leqslant \pi$ such that

$$B = R(\mathbf{i},\alpha)\, R(\mathbf{j},\beta)\, R(\mathbf{i},\gamma).$$

[Hint: note that $R(\mathbf{i},-\alpha)B\mathbf{i}$ must be a linear combination of $\mathbf{i}$ and $\mathbf{k}$.]

4. Let

$$A(t) = \frac{1}{9}\begin{pmatrix} 4+5\cos t & -4+4\cos t+3\sin t & 2-2\cos t+6\sin t \\ -4+4\cos t-3\sin t & 4+5\cos t & -2+2\cos t+6\sin t \\ 2-2\cos t-6\sin t & -2+2\cos t-6\sin t & 1+8\cos t \end{pmatrix}.$$

Given that $A(t)$ is an orthogonal matrix, find its angular velocity.

5. Let

$$\mathbf{e}_r = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

where θ and ϕ are functions of time t. (i) Show that $\mathbf{e}_r$ has unit length and that

$$\dot{\mathbf{e}}_r = \dot{\theta}\mathbf{e}_\theta + \dot{\phi}\sin\theta\,\mathbf{e}_\phi$$

for two unit vectors $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$ which you should determine. Find similar expressions for $\dot{\mathbf{e}}_\theta$ and $\dot{\mathbf{e}}_\phi$. (ii) Show that $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_\phi$ form a right-handed orthonormal basis. (iii) Find the angular velocity $\boldsymbol{\omega}$, in terms of $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_\phi$, such that

$$\dot{\mathbf{e}}_r = \boldsymbol{\omega} \wedge \mathbf{e}_r, \qquad \dot{\mathbf{e}}_\theta = \boldsymbol{\omega} \wedge \mathbf{e}_\theta, \qquad \dot{\mathbf{e}}_\phi = \boldsymbol{\omega} \wedge \mathbf{e}_\phi.$$

6. (*Optional*) The matrix $A(t)$ below is orthogonal and has determinant 1. [You do not need to verify this.]

$$A(t) = \begin{pmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{pmatrix}.$$

(i) Show that $A'(t) = WA$ where W is a constant matrix such that $W^T = -W$.

(ii) Determine W^2 and hence show that $A(t) = e^{Wt}$ where the exponential of a square matrix is defined by

$$e^X = I + X + X^2/2! + X^3/3! + \cdots.$$

(iii) Show, in general, that if X is an anti-symmetric matrix (that is $X^T = -X$) then e^X is an orthogonal matrix.