**GEOMETRY – SHEET 7 – Surface Area.**
(Exercises on lectures in Week 8)

**1.** Let $\mathbf{r}$ be a parametrization of a subset of the $xy$-plane in $\mathbb{R}^3$ so that

$$\mathbf{r}(u,v) = (x(u,v), y(u,v), 0), \qquad (u,v) \in U.$$

Determine $\mathbf{r}_u \wedge \mathbf{r}_v$ and show that the area of $\mathbf{r}(U)$ equals

$$\iint_U \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \mathrm{d}u\,\mathrm{d}v.$$

**2.** Part of a catenoid is formed by rotating the graph of $y = \cosh x$ (where $a \leqslant x \leqslant b$) about the $x$-axis. Calculate the area $A$ of this surface by using the formula

$$A = 2\pi \int_{x=a}^{x=b} y\,\mathrm{d}s.$$

**3.** Let $0 < \alpha < \pi/2$ and let $S_\alpha$ be the cap of the unit sphere $x^2+y^2+z^2 = 1$ parametrized as

$$\mathbf{r}(\theta,\phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \qquad 0<\theta<\alpha,\ 0<\phi<2\pi.$$

(i) Determine $\mathbf{r}_\theta \wedge \mathbf{r}_\phi$ and show that the surface area of $S_\alpha$ equals $2\pi(1-\cos\alpha)$.

(ii) Rederive the result of (i) by considering the cap as part of the graph $z = \sqrt{1-x^2-y^2}$ and using the formula for a graph's surface area.

**4.** Let $\theta$ and $\phi$ denote spherical polar co-ordinates on the unit sphere $x^2+y^2+z^2=1$. The *Albers equal-area conic projection* is defined by

$$x = \frac{1}{n}\sqrt{C-2n\cos\theta}\,\sin n\phi, \qquad y = \rho_0 - \frac{1}{n}\sqrt{C-2n\cos\theta}\,\cos n\phi,$$

where $n, C, \rho_0$ are constants. Describe the image in the plane of a latitude ($\theta =$ const.) and a meridian ($\phi =$ const.) Determine the Jacobian $|\partial(x,y)/\partial(\phi,\theta)|$ and explain why this means the projection preserves area.

**5.** (i) Let $\mathbf{r}(u,v)$ be a parametrization of a surface in $\mathbb{R}^3$ and $\gamma(t) = \mathbf{r}(u(t),v(t))$. Prove the following:

$$\gamma'(t) = u'\mathbf{r}_u + v'\mathbf{r}_v, \qquad |\gamma'(t)|^2 = E(u')^2 + 2Fu'v' + G(v')^2, \qquad |\mathbf{r}_u \wedge \mathbf{r}_v|^2 = EG - F^2,$$

where $E = \mathbf{r}_u\cdot\mathbf{r}_u$, $F = \mathbf{r}_u\cdot\mathbf{r}_v$, $G = \mathbf{r}_v\cdot\mathbf{r}_v$.

(ii) The *flat torus* in $\mathbb{R}^4$ can be parametrized as

$$\mathbf{r}(\theta,\phi) = (\cos\theta, \sin\theta, \cos\phi, \sin\phi), \qquad (\theta,\phi) \in U = (0,2\pi)\times(0,2\pi).$$

Let $\gamma(t) = (\theta(t), \phi(t))$ be a curve in $U$ where $a \leqslant t \leqslant b$. Show that the curve $\gamma$, and its image $\mathbf{r}(\gamma)$ in the flat torus, have the same length.

As there is no vector product in $\mathbb{R}^4$ we cannot use our current formulae to work out the surface area of the flat torus. What do you think its surface area equals?

**6.** (*Optional*) Consider the second-order partial differential equation

$$Az_{xx} + Bz_{xy} + Cz_{yy} = 0$$

where $A, B, C$ are constants. Show that there is a value of $\theta$ such that the substitution

$$X = x\cos\theta + y\sin\theta, \qquad Y = -x\sin\theta + y\cos\theta,$$

turns the above equation into one of the form $z_{XX} + kz_{YY} = 0$. When is $k = 0$?

**7.** (*Optional*) Let $0 < a \leqslant b \leqslant c < \pi/2$ and let $A = (0,0,1)$ and $B = (\sin c, 0, \cos c)$ be points on the unit sphere.

(i) Show that there is a point $C$ which is at a distance $b$ from $A$ and at a distance $a$ from $B$ if and only if $a + b \geqslant c$.

[Hint: every point at a distance $b$ from $A$ has the form $(\sin b\cos\phi, \sin b\sin\phi, \cos b)$ for some $\phi$.]

(ii) Prove the *spherical cosine rule* which states that for a spherical triangle with sides $a, b, c$ opposite angles $\alpha, \beta, \gamma$

$$\cos a = \cos b\cos c + \sin b\sin c\cos\alpha.$$

Show that when $a, b, c$ are small enough, that we can ignore terms of order 3 and above, so that the approximations $\cos x \approx 1 - x^2/2$ and $\sin x \approx x$ apply, then the spherical cosine rule approximates the usual cosine rule.