Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, $\mathbf{e}_4$ denote the standard basis of $\mathbb{R}^4$. Suppose for a contradiction that $\wedge$ is a vector product on $\mathbb{R}^4$ with properties (a)–(d) as described in lectures.

As $\mathbf{e}_1 \wedge \mathbf{e}_2$ must be perpendicular to $\mathbf{e}_1$ and $\mathbf{e}_2$ and $\mathbf{e}_1 \wedge \mathbf{e}_3$ must be perpendicular to $\mathbf{e}_1$ and $\mathbf{e}_3$ then

$$\mathbf{e}_1 \wedge \mathbf{e}_2 = \alpha \mathbf{e}_3 + \beta \mathbf{e}_4, \qquad \mathbf{e}_1 \wedge \mathbf{e}_3 = \gamma \mathbf{e}_2 + \delta \mathbf{e}_4$$

for some $\alpha, \beta, \gamma, \delta$. Further by (d) we know

$$\alpha^2 + \beta^2 = 1 = \gamma^2 + \delta^2.$$

Now the vector product

$$\mathbf{e}_1 \wedge \left( \frac{\mathbf{e}_2 + \mathbf{e}_3}{\sqrt{2}} \right) = \frac{\gamma \mathbf{e}_2 + \alpha \mathbf{e}_3 + (\beta + \delta)\mathbf{e}_4}{\sqrt{2}}$$

must be of unit size by (d) and perpendicular to $\mathbf{e}_2 + \mathbf{e}_3$. It follows from the latter that

$$\gamma + \alpha = 0$$

and also that

$$\gamma^2 + \alpha^2 + (\beta + \delta)^2 = 2.$$

As we already know that $\alpha^2 + \beta^2 = 1 = \gamma^2 + \delta^2$ then it follows that $\beta\delta = 0$.
Case (i): $\beta = 0$. Then $\alpha = \pm 1$ and $\gamma = \mp 1$. It then follows that $\delta = 0$ as well.
Case (ii): $\delta = 0$. Then $\gamma = \pm 1$ and $\alpha = \mp 1$. It then follows that $\beta = 0$ as well.
So in any case we have $\beta = \delta = 0$.

We now consider $\mathbf{e}_1 \wedge \mathbf{e}_4$. We likewise have

$$\mathbf{e}_1 \wedge \mathbf{e}_4 = A\mathbf{e}_2 + B\mathbf{e}_3$$

for some $A$, $B$ with $A^2 + B^2 = 1$ and then

$$\mathbf{e}_1 \wedge \left( \frac{\mathbf{e}_2 + \mathbf{e}_4}{\sqrt{2}} \right) = \frac{A\mathbf{e}_2 + (\alpha + B)\mathbf{e}_3}{\sqrt{2}}.$$

As this is perpendicular to $\mathbf{e}_2 + \mathbf{e}_4$ then $A = 0$ and then $B^2 = 1$. We have then

$$\mathbf{e}_1 \wedge \mathbf{e}_2 = \alpha \mathbf{e}_3, \qquad \mathbf{e}_1 \wedge \mathbf{e}_4 = B\mathbf{e}_3$$

with $\alpha^2 = B^2 = 1$. And so

$$\mathbf{e}_1 \wedge \left( \frac{\alpha \mathbf{e}_2 - B\mathbf{e}_4}{\sqrt{2}} \right) = \frac{(\alpha^2 - B^2)\mathbf{e}_3}{\sqrt{2}} = \mathbf{0}.$$

This contradicts (d) as $\mathbf{e}_1$ and $(\alpha \mathbf{e}_2 - B\mathbf{e}_4)/\sqrt{2}$ are two perpendicular unit length vectors and so their product should have unit length.