Part A Graph Theory

Marc Lackenby

Trinity Term 2022

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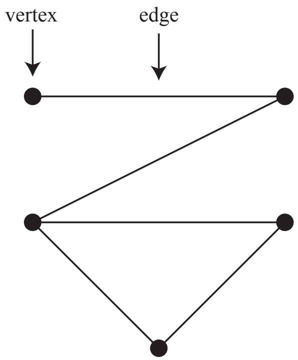

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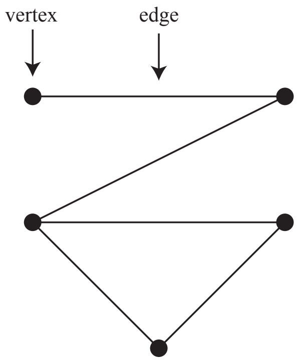

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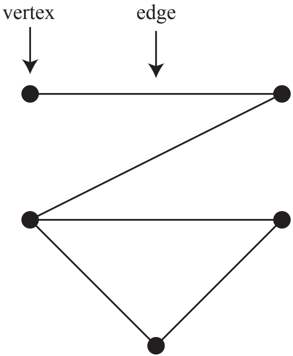
We will give a formal definition shortly.



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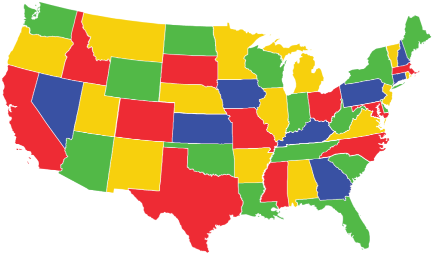

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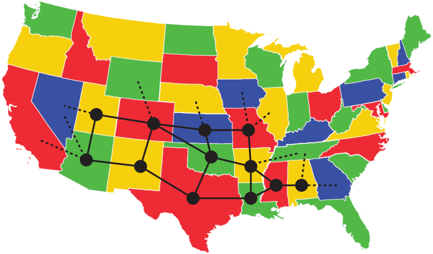

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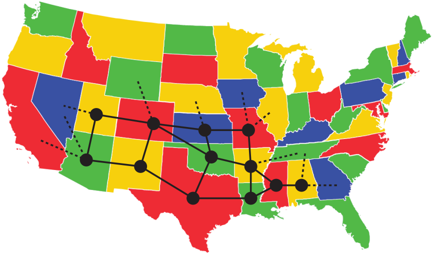

This was proved by Appel and Haken in 1976, using a controversial computer-assisted proof.

Connectedness

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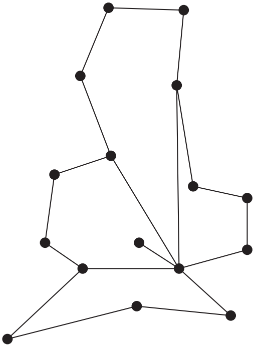

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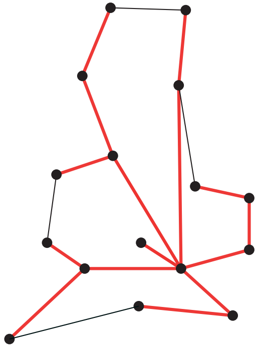

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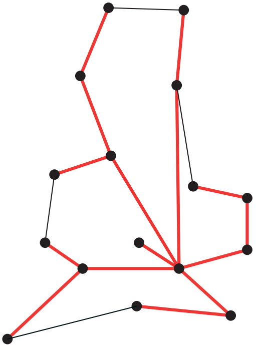

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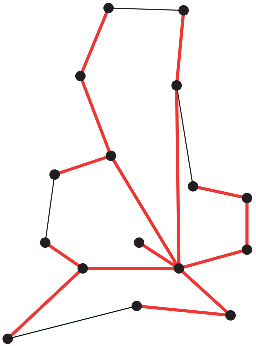

Let us make some definitions and formulate this problem mathematically.

Definitions

A graph $G = (V(G), E(G))$ consists of two sets:

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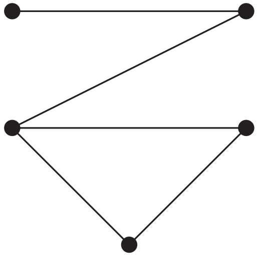

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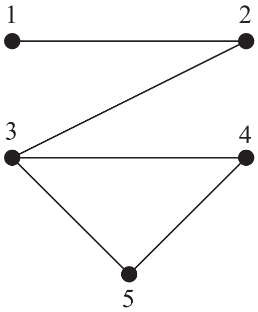

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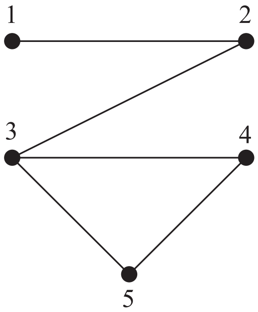

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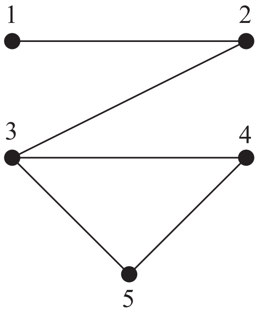
We will always assume without further comment that

$|V(G)|$ is finite.



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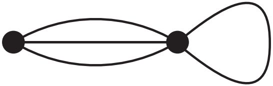

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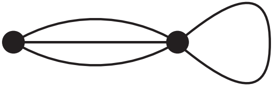

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We write $uv = \{u, v\} = vu$ for the (unordered) pair representing an edge between u and v.

Connectedness

Walks, paths and cycles

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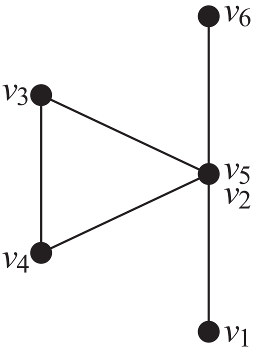

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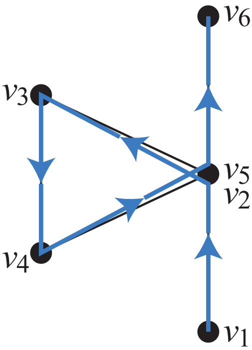

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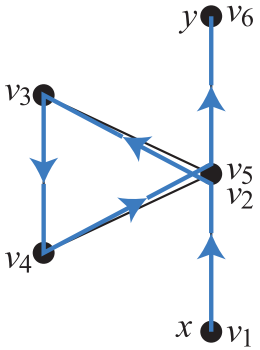

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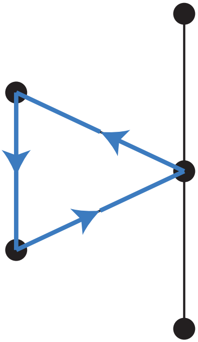

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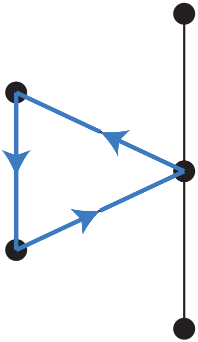

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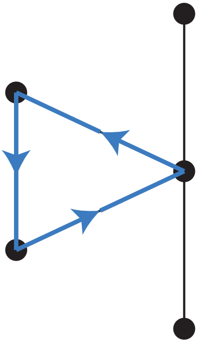
We also regard paths and cycles as subgraphs of G.



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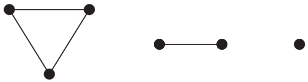

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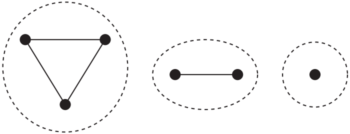

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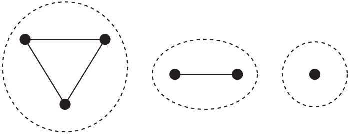

Clearly this forms an equivalence relation and the partition of $V(G)$ into equivalence classes expresses G as a union of disjoint connected graphs called its *components*.

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Our task:

Find $S \subseteq E(G)$ with minimum possible $c(S)$ such that $(V(G), S)$ is a connected graph.

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We are interested in 'efficient algorithms'. We will not define this concept precisely in this course, but it will be exemplified by the algorithms that we present.

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This motivates the next section.

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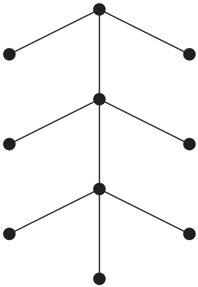

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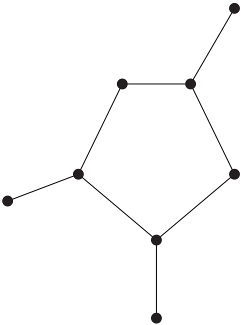

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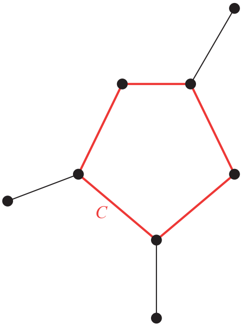

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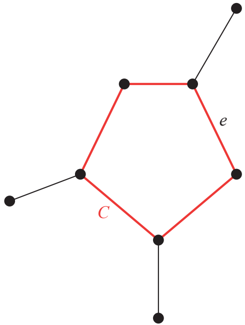

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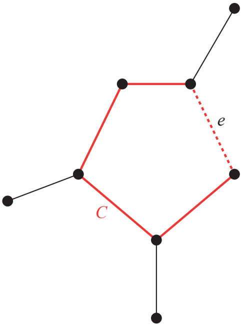

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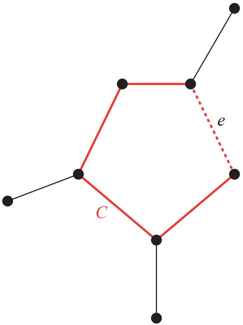

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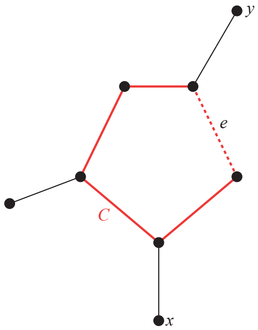

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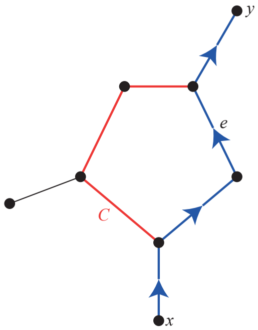

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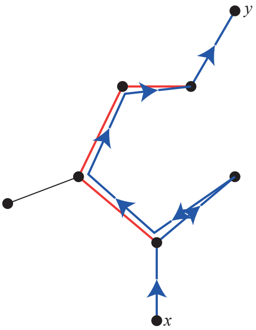
Replacing any use of e in W by P gives an xy-walk in $G - e$. Thus $G - e$ is connected, contradiction. □



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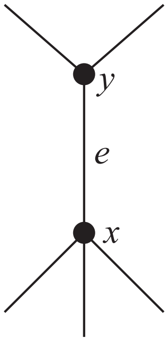

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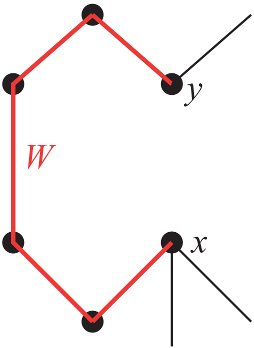

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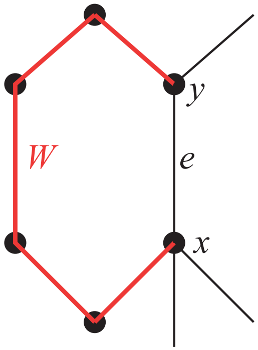

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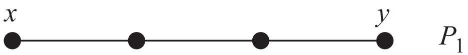

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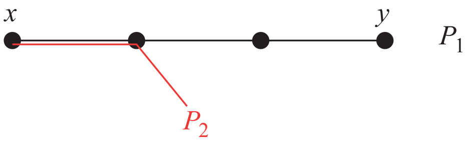

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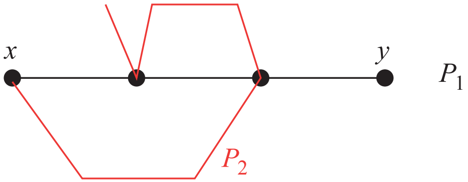

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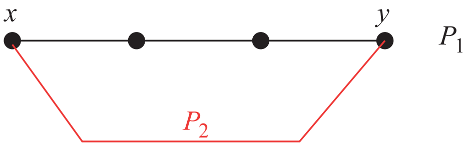

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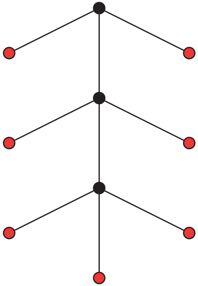

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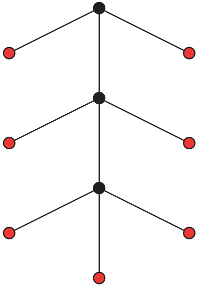

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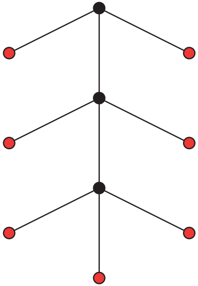

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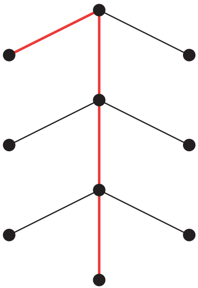

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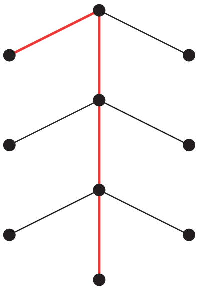

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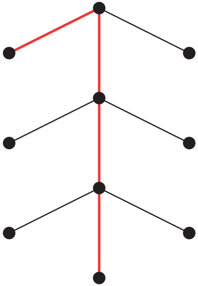

Removing a leaf from a tree

Given $v \in V(G)$, let $G - v$ be the graph with
$V(G - v) = V(G) \setminus \{v\}$ and
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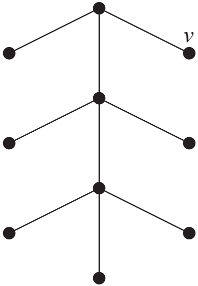

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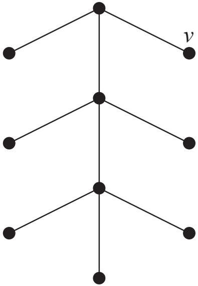

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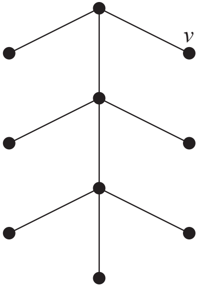

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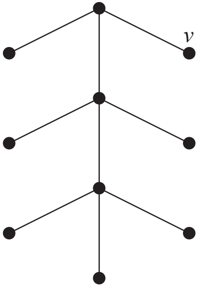

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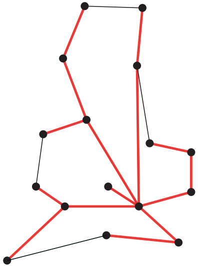

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Conversely, suppose that G is connected and has $n-1$ edges. Let H be a spanning tree of G. Then H has $n-1$ edges by Lemma 6, so $H = G$, so G is a tree. $\square$

Minimum Cost Spanning Trees

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A spanning tree T for G has minimum cost if any other spanning tree T' satisfies $c(T') \geq c(T)$.

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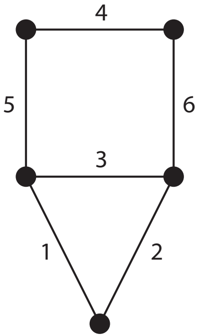

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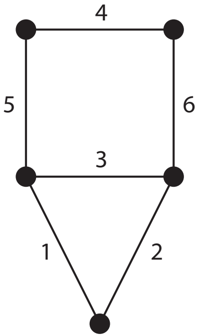

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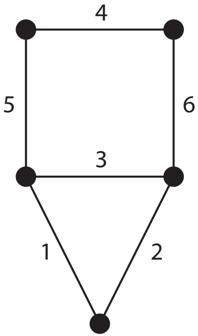

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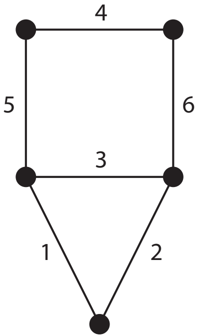

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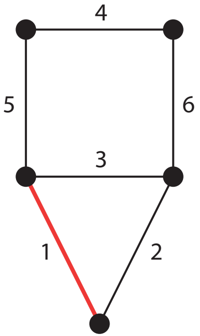

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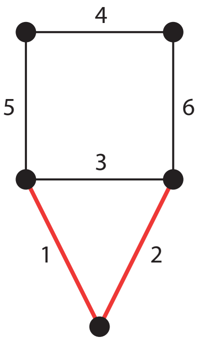

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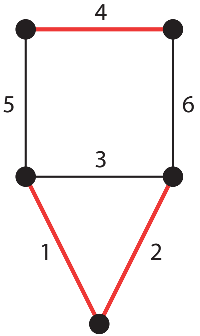

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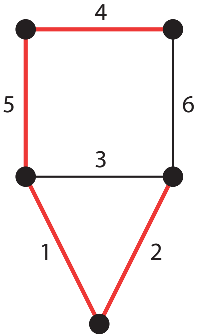

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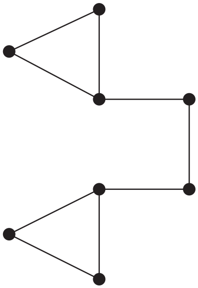

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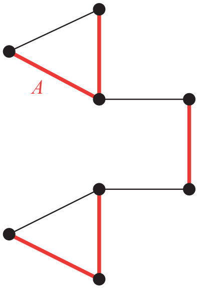

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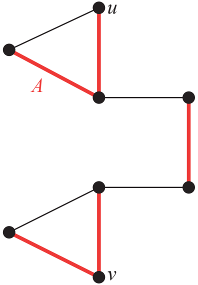

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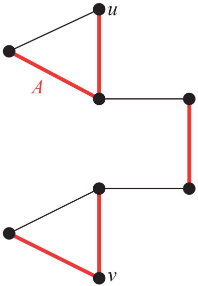

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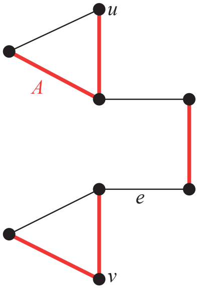

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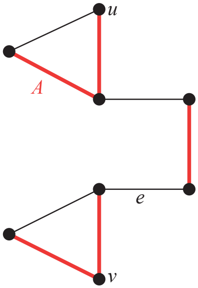

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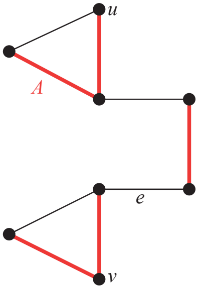

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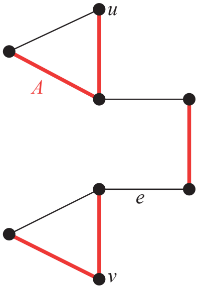

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Note that $(*)$ will suffice to prove the theorem, as when we apply it to $A_i = A$ we will have $A \subseteq B$ for some $B \in \mathcal{M}$ and so $|A| = |B|$ by Lemma 6, and so $A = B \in \mathcal{M}$.

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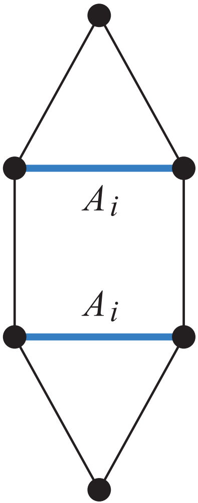

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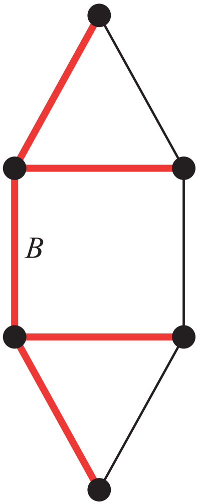

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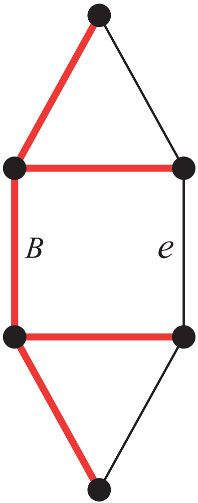
Consider $A_{i+1} = A_i \cup \{e\}$ given by the algorithm. We need to find $B' \in \mathcal{M}$ with $A_{i+1} \subseteq B'$. We can assume $e \notin B$, otherwise we could take $B' = B$.



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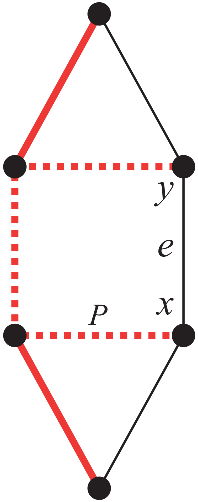

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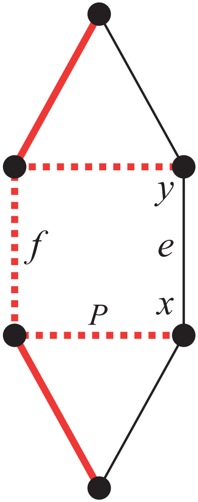

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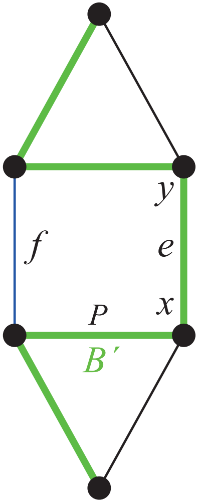

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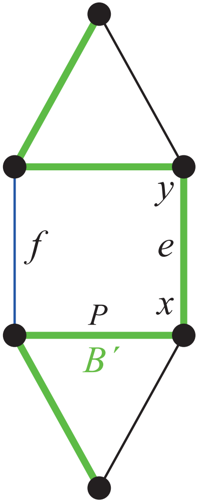
To finish the proof we need to show that

1. $A_{i+1} \subseteq B'$,
2. $(V(G), B')$ is a spanning tree, and
3. $c(B') \leq c(B)$.



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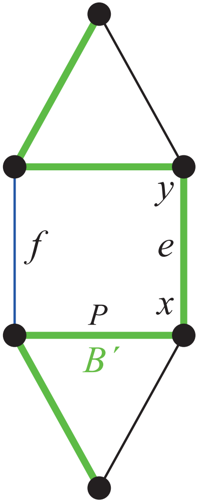

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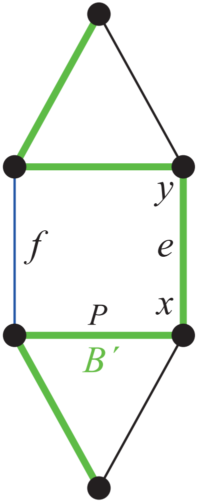

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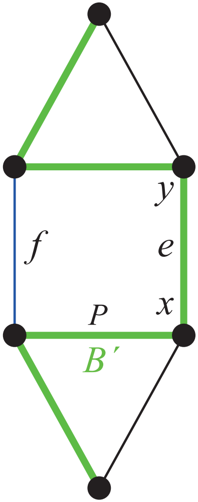

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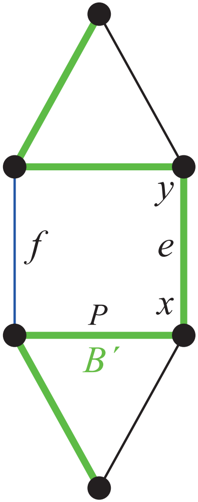
Note that B' is connected, for the following reason. Any two vertices in $V(G)$ are joined by a path in B. Replace each occurence of f in this path by $C \setminus \{f\}$. Also B' has $|V(G)| - 1$ edges. So it is a spanning tree by Lemma 7.



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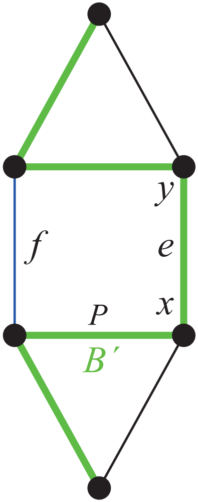

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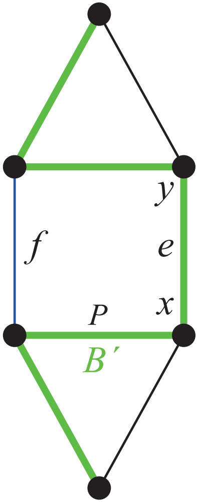

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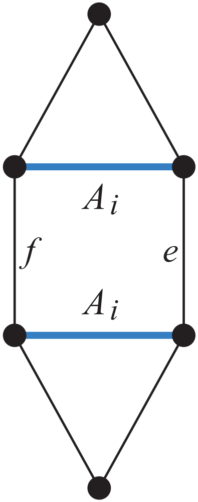

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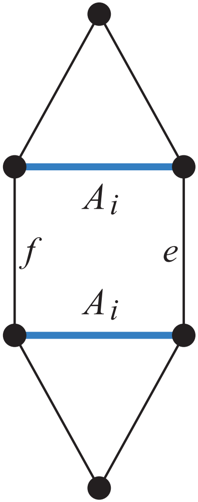

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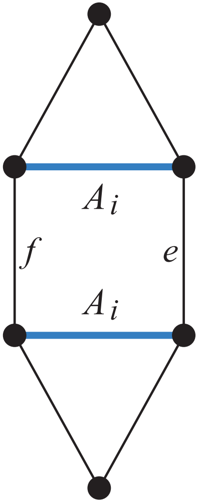
This finishes the proof of the inductive step of $(*)$, and so of the theorem. □



The number of steps of Kruskal's algorithm

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To make this question mathematically precise would take us far afield (we would need to define a model of computation). In this course, we will take the intuitive approach of estimating the number of 'steps' taken by an algorithm, where a 'step' should be a 'simple' operation.

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Here 'running time' could be measured in any units, say milliseconds on your favourite computer, as changing the units or using a different computer will just replace C by a different constant.

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A smarter implementation is to start by making a list of all edges ordered by cost, cheapest first. Then at each step we go through the list from the start, discarding edges that make a cycle until we find the first edge which can be added.

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This gives a running time that is 'roughly comparable' with the number of edges, which is essentially best possible.

Euler tours

The bridges of Königsberg

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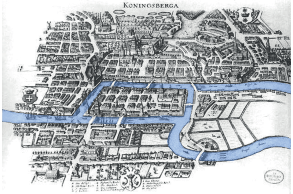

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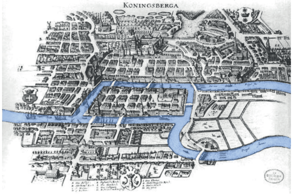

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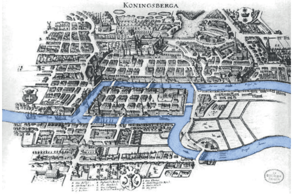

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Let W be a walk in a graph G. We call W an *Euler trail* if every edge of G appears exactly once in W.

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Here we will only solve the problem of finding an Euler tour; the solution of the Euler trail problem can be deduced (see exercise sheet 1).

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In fact, we will show that we can find an Euler tour efficiently, using the following algorithm.

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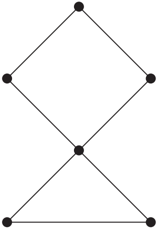

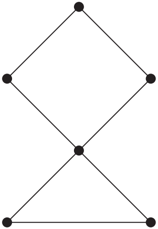

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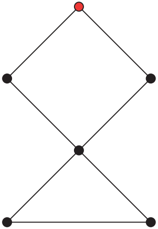

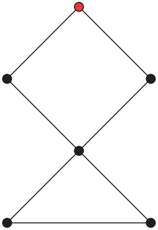

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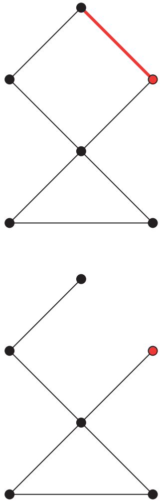

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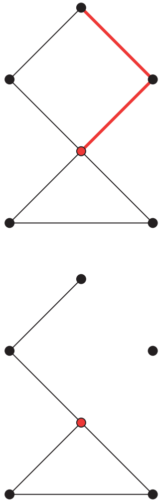

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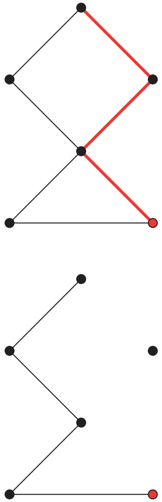

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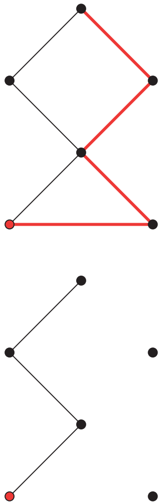

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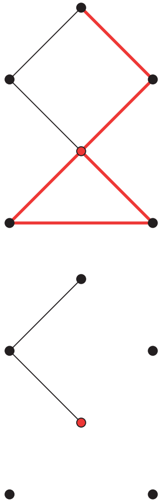

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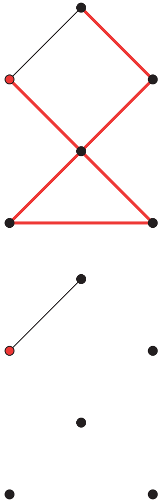

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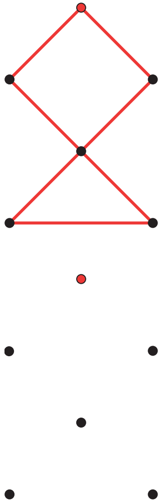

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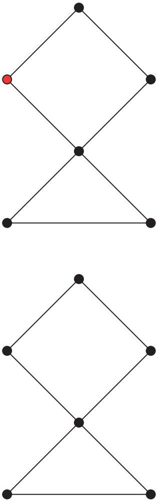

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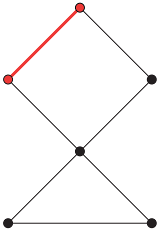

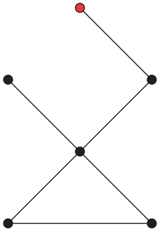

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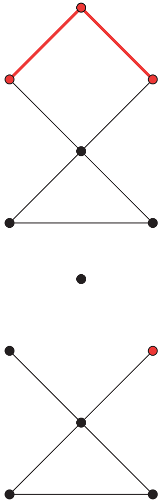

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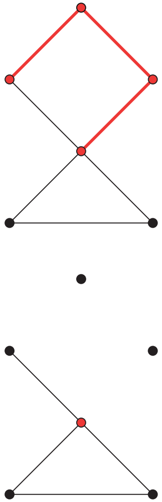

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Proof. Since every edge has two endpoints,

$$\sum_{v \in V(G)} d(v) = 2|E(G)|.$$

Therefore, in the sum, there must be an even number of occurrences of $d(v)$ for which $d(v)$ is odd. $\square$

Fleury's algorithm produces an Euler tour

Note first that at each stage of the algorithm, either there are two vertices of odd degree (the initial vertex u and the current one) or there are no vertices of odd degree.

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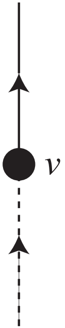

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If the degree of v in H is one, then we can continue the walk.

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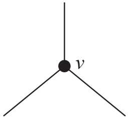

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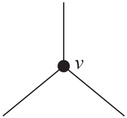

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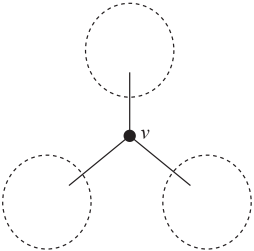

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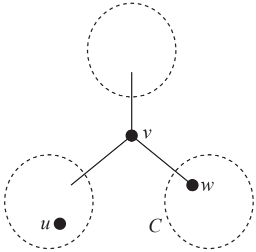

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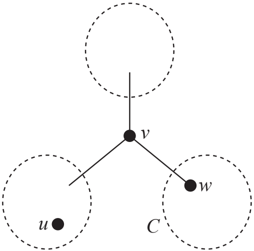
But then w is the only vertex of odd degree in C, which is impossible by Lemma 10. □



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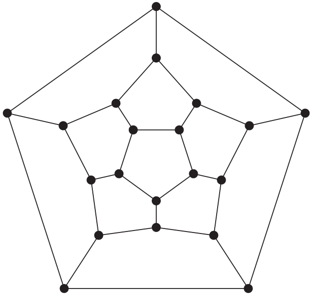

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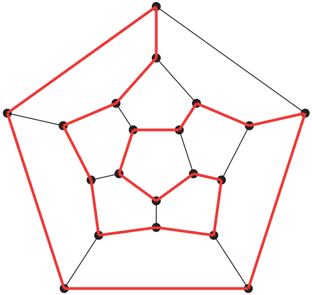

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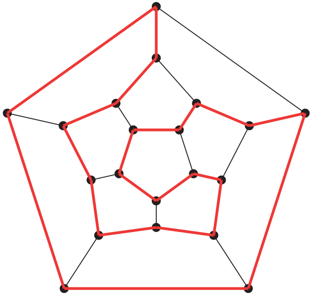

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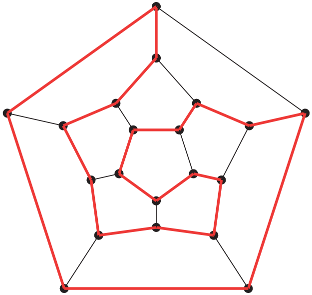

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But to discuss this conjecture would take us too far afield.

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Corollary 12. If G is connected with n vertices and for every vertex v, $d(v) \geq n/2$, then G is Hamiltonian.

The length of paths and cycles

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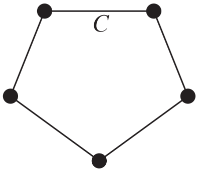

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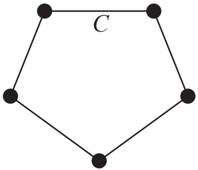

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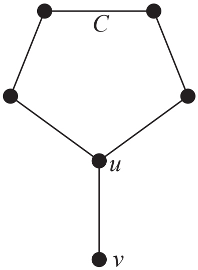

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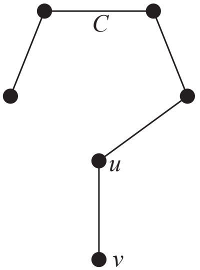
Proof. Let C be a longest cycle, with length ℓ. Since G is non-Hamiltonian, there is some vertex not in C. Since G is connected, there is therefore some edge uv with one endpoint u in C and one endpoint v not in C. Removing an edge of C incident to u and adding uv gives a path of length ℓ. □



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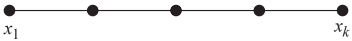

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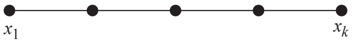

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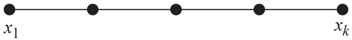

So by Lemma 13, G does not have a cycle of length k. So x_1 and x_k are not adjacent.

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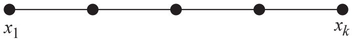

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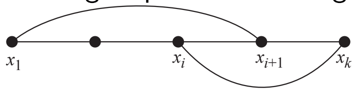

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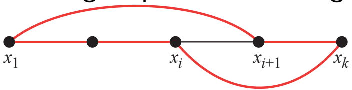

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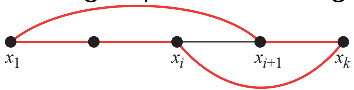

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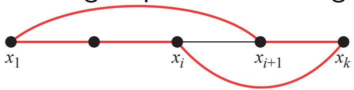

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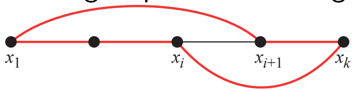

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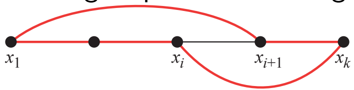

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