Prelims Introductory Calculus MT 2018: Sheet 4

1. Given that
$$xs^2 + yt^2 = 1 \quad \text{and} \quad x^2 s + y^2 t = xy - 4,$$
where $x = x(s,t)$ and $y = y(s,t)$, find $x_s,\ x_t,\ y_s$ and y_t at the point $(x,y,s,t) = (1,-3,2,-1)$.

2. Given that
$$\begin{aligned} z &= x^2 + xy, \\ x^2 + y^3 &= st + 5, \\ x^3 - y^2 &= s^2 + t^2, \end{aligned}$$
where $x = x(s,t)$ and $y = y(s,t)$, find expressions for $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$.

Evaluate these expressions at the point $(x,y,s,t) = (3,1,1,5)$.

3. Let $t > 0$. Verify that
$$T(x,t) = \frac{A}{\sqrt{t}} \exp\left(\frac{-x^2}{4\kappa t}\right)$$
is a solution of the heat equation
$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}.$$
Sketch T as a function of x at two different times t.

4. Find general solutions $f(x,y,z)$ of the following PDES:
$$\text{(a)} \quad \frac{\partial^3 f}{\partial z^3} = 0, \qquad \text{(b)} \quad \frac{\partial^3 f}{\partial x \partial y \partial z} = 0.$$

5. Find general solutions $u(x,y)$ of the following PDES:
$$\text{(a)} \quad y\frac{\partial u}{\partial y} = u, \qquad \text{(b)} \quad \frac{\partial u}{\partial x} = 2xyu.$$

6. Use the change of variable $u(x,t) = e^{\beta t} g(x)$ to find solutions of the equation
$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial t} + \frac{\partial^2 u}{\partial t^2} = 0.$$

7. Find separable solutions $z(x,y) = X(x)Y(y)$ of the following PDEs:
$$\text{(a)} \quad \frac{\partial z}{\partial y} = y\frac{\partial z}{\partial x}, \qquad \text{(b)} \quad x\frac{\partial z}{\partial x} = z + y\frac{\partial z}{\partial y}.$$