Prelims Introductory Calculus MT 2018: Sheet 6

1. Evaluate

$$\text{(a)} \int_{x=0}^{x=1} \int_{y=0}^{y=3-3x} \mathrm{d}y\ \mathrm{d}x, \quad \text{(b)} \int_{y=0}^{y=1} \int_{x=0}^{x=\sqrt{1-y^2}} \mathrm{d}x\ \mathrm{d}y, \quad \text{(c)} \int_{x=0}^{x=4} \int_{y=0}^{y=\sqrt{x}} \mathrm{d}y\ \mathrm{d}x.$$

   In each case, sketch the area of integration.

2. Use a double integral to find the area of the region bounded by the curves $y^2 = 4x$ and $2x + y = 4$.

3. Let $a > 0$. Sketch the curve

$$x^{2/3} + y^{2/3} = a^{2/3}$$

   in the quadrant $x, y > 0$.

   The variables u and v are given in terms of x and y by

$$x = u \cos^3 v, \quad y = u \sin^3 v.$$

   What is the equation of the curve in terms of the new coordinates u and v?

   Calculate the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$ and hence find the area of the region bounded by the curve and the positive x- and y-axes.

4. Let $a > 0$. The curve with polar equation

$$r = a(1 + \cos\theta), \qquad 0 \leq \theta < 2\pi$$

   is called a cardioid. Sketch the curve and, using a double integral, show that the area bounded by it equals $\frac{3}{2}\pi a^2$.

5. Calculate the scalar line integral of the vector field

$$\mathbf{F}(\mathbf{r}) = (2xy^2 - 3yz + 1, 2yx^2 - 3xz, -3xy)$$

   along the path consisting of the straight-line segment joining the origin to the point (a, b, c).

6. Find the length of the curve given by

$$x = t, \quad y = -\ln(\cos t), \qquad 0 \leq t \leq \pi/4.$$