Prelims Introductory Calculus MT 2016: Sheet 5

1. Calculate the Jacobians $\dfrac{\partial(u,v)}{\partial(x,y)}$ and $\dfrac{\partial(x,y)}{\partial(u,v)}$, and verify that $\dfrac{\partial(u,v)}{\partial(x,y)}\dfrac{\partial(x,y)}{\partial(u,v)} = 1$, in each of the following cases:

   (i) $u = x + y, \quad v = \dfrac{y}{x}$; (ii) $u = \dfrac{x^2}{y}, \quad v = \dfrac{y^2}{x}$.

2. The variables u and v are given by

$$u = x^2 - xy, \qquad v = y^2 + xy$$

   for all real x and y. By finding an appropriate Jacobian matrix, calculate the partial derivatives x_u, x_v, y_u and y_v in terms of x and y only. State the values of x and y for which your results are valid.

3. Recall the definition of parabolic coordinates:

$$x = \frac{1}{2}(u^2 - v^2), \qquad y = uv.$$

   Show that Laplace's equation in Cartesian coordinates, that is

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = 0,$$

   transforms into the same equation in parabolic coordinates.

4. In the partial differential equation

$$\frac{\partial^2 z}{\partial x^2} - 5\frac{\partial^2 z}{\partial x \partial y} + 6\frac{\partial^2 z}{\partial y^2} = 0,$$

   make the change of variables $s = y + 2x,\ t = y + 3x$ and show that the PDE becomes

$$\frac{\partial^2 z}{\partial s \partial t} = 0.$$

   Hence solve the original PDE.

5. Laplace's equation in three dimensions is given in Cartesian coordinates by

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2} = 0.$$

   For spherical polar coordinates defined by

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta,$$

   show that Laplace's equation becomes

$$\frac{\partial^2 F}{\partial r^2} + \frac{2}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial F}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 F}{\partial \phi^2} = 0.$$

   Hint: Use the inverse transformation as given in the lecture notes.