**CFT problems**

**Sheet 1 - Chapters 1 and 2**

**1.** For each of the following theories compute the canonical scaling dimensions of the fields and write down all the relevant and marginal operators (Remember: the operators should not have free Lorentz indices). Furthermore, give the scaling dimension of the coupling constants appearing in the Lagrangian.

**Two scalar fields in $d = 3$ and $d = 4$**

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_1)^2 + \frac{1}{2}(\partial_\mu \phi_2)^2 + \frac{\lambda}{4!}\left(\phi_1^4 + \phi_2^4\right) + \frac{2\rho}{4!}\phi_1^2\phi_2^2$$

**Dirac Lagrangian in $d = 4$**

$$\mathcal{L} = \bar{\psi} i \Gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi$$

**Coleman-Weinberg model in $d = 4$**

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + (D_\mu \phi)^2 + m^2\phi^2 + \lambda\phi^4$$

where the field $\phi$ is real, the signature is Euclidean and $D_\mu\phi = (\partial_\mu + eA_\mu)\phi$.

**2.** The beta function describes the dependence of a coupling parameter $g$ on the energy scale $\mu$ of a given physical process:

$$\beta(g) = \mu\frac{\partial g}{\partial \mu}$$

such that at fixed points of the RG flow beta functions should vanish. Consider a non-abelian gauge theory with gauge group $SU(N_c)$ and $N_f$ Dirac fermions in the fundamental representation of the gauge group. For large $N_c, N_f$ its perturbative beta function is given by

$$\beta(g) = -\frac{1}{16\pi^2}\frac{1}{3}(11N_c - 2N_f)g^3 - \frac{1}{(16\pi^2)^2}\left(\frac{34}{3}N_c^2 - \frac{13}{3}N_f N_c\right)g^5 + \cdots$$

Determine for which region of the parameters $N_f, N_c$ there are fixed point consistent with perturbation theory.

**3.** Given the finite conformal transformations check explicitly that a special conformal transformation is equivalent to an inversion followed by a translation followed by another inversion. How are the parameters $a^\mu$ and $b^\mu$ related?

**4.** From the action of the conformal generators acting on functions, verify all the commutation relations of the conformal algebra except those between two Lorentz rotations.

**5. Quadratic Casimir**. Consider the following quadratic operator

$$\mathcal{C}^{(2)} = L_{\mu\nu}L^{\mu\nu} + \alpha P_\mu K^\mu + \beta K_\mu P^\mu + \gamma D^2$$

and fix the coefficients $\alpha, \beta, \gamma$ such that it commutes with all the generators of the conformal algebra.

**6.** Following the method of the lectures, derive $[K_\mu, \phi_\alpha(x)]$ for a primary *scalar* operator.

**Sheet 2 - Chapters 3 and 4**

**1.** Consider the canonical energy-momentum tensor for the free boson in $d > 2$. Find an improvement term which makes it classically traceless without spoiling classical conservation.

*Hint: Note that the index structure suggests a general ansatz of the form:*

$$(\alpha \eta^{\mu\nu} \partial_\rho \partial^\rho + \beta \partial^\mu \partial^\nu) f(x)$$

**2.** Consider two dimensional Liouville theory with the following Lagrangian density

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} m^2 e^\phi$$

write down the canonical energy momentum tensor and verify that it is conserved (on the equations of motion). Add a term such that it is also traceless, without spoiling conservation.

**3.** Prove the following property under special conformal transformations

$$|x'_i - x'_j| = \frac{|x_i - x_j|}{\gamma_i^{1/2} \gamma_j^{1/2}}$$

where $\gamma_i = 1 - 2b \cdot x_i + b^2 x_i^2$.

**4.** Consider the inversion tensor $I_{\mu\nu}(x) = \eta_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2}$. Show that under inversions

$$I_{\mu\alpha}(x) I^{\alpha\beta}(x - y) I_{\beta\nu}(y) = I_{\mu\nu}(x' - y')$$

where $(x')^\mu = x^\mu / x^2$ and $(y')^\mu = y^\mu / y^2$.

**5.** Consider the OPE $\phi_1(x)\phi_2(0)$ and the contribution to this OPE from a single primary $\phi_\Delta(0)$ plus all its tower of descendants:

$$\phi_1(x)\phi_2(0)|0\rangle = \frac{const}{|x|^{\Delta_1 + \Delta_2 - \Delta}} \left(\phi_\Delta(0) + \alpha x^\mu \partial_\mu \phi_\Delta(0) + \cdots\right)|0\rangle$$

In the lectures it has been shown how to fix $\alpha$ by acting on both sides with $K_\mu$. By following the same idea compute the next orders in the above expansion, quadratic in $x$.

**6.** Rederive the results above by considering appropriate two and three point functions in a conformal field theory, following the "practical method" described in the lecture notes.

**Sheet 3 - CFT in two dimensions**

**1.** a.- Given four point in the complex plane $z_1, \cdots z_4$ show that the cross-ratio $\eta$ defined in the lectures is invariant under global conformal transformations.

b.- Find a global transformation that maps the points $(0, i, 2)$ to the points $(0, 1, \infty)$.

**2.** Consider a free scalar field in two dimensions $\varphi(x)$ and the operator $\mathcal{O}_\alpha =: e^{i\alpha\varphi(z)} :$, where $\alpha$ is a real constant. Focusing only in its holomorphic dependence, compute the OPE of this operator with the stress tensor and verify that it is a primary operator of a given weight that you should compute. [*Note: The normal ordering symbol is meant to remind us not to Wick contract two scalar fields within the operator*]. Show that the two point function of such operators behaves as it should.

**3.** a.- Calculate the four-point function $\langle \partial\varphi\partial\varphi\partial\varphi\partial\varphi \rangle$ for the free two-dimensional boson, using Wick contraction. Compare it with the general expression given in the lectures and determine the function $g(\eta)$ in this case.

b.- Calculate now the correlator $\langle T(z)\partial\varphi\partial\varphi\partial\varphi\partial\varphi \rangle$, where $T(z)$ is the holomorphic stress tensor given in the lectures, using Wick contraction. Verify the conformal ward identities for this case.

**4.** Show that the Schwartzian derivative vanishes when restricted to global conformal transformations.

**5.** Given a Virasoro primary $|h\rangle$ such that

$$L_0|h\rangle = h|h\rangle, \quad \langle h|h\rangle = 1$$

Compute the inner products between all level two descendants and their conjugates.

**6. Identity Virasoro conformal block**

Consider two identical operators of conformal weight $(h, \bar{h})$ such that they are canonically normalized

$$\langle \phi_{h,\bar{h}}(z, \bar{z})\phi_{h,\bar{h}}(0) \rangle = \frac{1}{z^{2h}\bar{z}^{2\bar{h}}}$$

Consider the OPE expansion (6.25) in the lecture notes, and focus in the identity operator plus its Virasoro descendants.

a.-Compute the OPE coefficients $C_{12}^{Id,(k,\bar{k})}$ up to level two.

b.- Use the result of part $a$ to compute the small $z$ expansion of the Virasoro conformal block for the identity operator.

**Sheet 4 - Minimal models**

**1.** Given a Virasoro primary $|h\rangle$ determine the conditions for $|h\rangle$ to have level three null descendants. Write explicitly the expression for the descendants in each case.

**2.** In the lectures we have derived a differential equation for a correlator involving $\phi(z)$

$$\langle \phi(z)\phi_{h_1}(z_1)\cdots\rangle$$

where $\phi(z)$ is a primary field with a level two null descendant.

a.- Verify that the equation is automatically satisfied for two point functions of primary operators.

b.- Consider now a three point function and derive the selection rules stated in the lectures.

**3.** Consider the critical Ising model introduced in the lectures, and the four point correlator of four identical operators $\epsilon(z,\bar{z})$, of conformal dimension $h = \bar{h} = 1/2$.

a.- Explain why conformal symmetry implies:

$$\langle \epsilon(z_1,\bar{z}_1)\cdots\epsilon(z_4,\bar{z}_4)\rangle = \frac{g(\eta,\bar{\eta})}{|z_{12}|^{4h}|z_{34}|^{4h}}$$

b.- Given that $\epsilon(z,\bar{z})$ admits a level two null descendant, write down a differential equation for $g(\eta,\bar{\eta})$. [you may focus in the holomorphic dependence only]

c.- Write down the full correlator as a linear combination of solutions to the equation above [reintroducing the anti-holomorphic dependence].

d.- What form do the crossing relations take? Find the most general expression for $g(\eta,\bar{\eta})$ consistent with the crossing relations and with part (c). Can you think how to fix $g(\eta,\bar{\eta})$ completely?

**4.** Write down the schematic decomposition of the result of problem 3 in terms of Virasoro conformal blocks. Verify that the small $z,\bar{z}$ behaviour for the identity conformal block of problem 3, and the one computed in the lecture notes,agree with the small $z,\bar{z}$ behaviour you obtained in problem six of the previous sheet.