

ANALYSIS I: Problem sheet 3

Sequences: Convergence, Infinite Limits, Complex Sequences

[Where needed you may assume properties of familiar functions, even though they have not been rigorously defined yet in the course.]

1. For each of the following choices of a_n , and for arbitrary $\varepsilon > 0$, find N such that $|a_n| < \varepsilon$ whenever $n \geq N$. [You only need to find a value of N that works and not necessarily the smallest such N .]

$$(i) \frac{1}{n^2 + 3}, \quad (ii) \frac{1}{n(n - \pi)}, \quad (iii) \frac{1}{\sqrt{5n - 1}}.$$

2. Use sandwiching arguments to prove that, for each of the following choices of a_n , the sequence (a_n) converges to 0:

$$(i) \frac{n + 1}{n^2 + n + 1}, \quad (ii) 2^{-n} \cos(n^2), \quad (iii) \sin \frac{1}{n}, \quad (iv) \begin{cases} 1/2^n & \text{if } n \text{ is prime,} \\ -1/3^n & \text{otherwise.} \end{cases}$$

3. (a) Prove that $\sqrt{n+1} - \sqrt{n} \rightarrow 0$ by using the identity $a - b = (\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})$ (for $a, b \in \mathbb{R}^{\geq 0}$).
- (b) Prove that $n^{1/n} \geq 1$ for $n = 1, 2, \dots$. Let $a_n = n^{1/n} - 1$. Prove, by applying the binomial theorem to $(1 + a_n)^n$, that

$$a_n \leq \sqrt{\frac{2}{n-1}} \quad \text{for } n > 1.$$

Deduce that $n^{1/n} \rightarrow 1$.

4. (a) Write down, carefully quantified, what it means for a real sequence (a_n) (i) to be convergent; (ii) to be bounded; (iii) to tend to infinity. Write down, carefully quantified, the negations of (i), (ii), (iii).
- (b) Formulate and prove an analogue of the Sandwiching Lemma applicable to real sequences which tend to infinity.
- (c) For each of the following choices of a_n decide whether or not the sequence tends to infinity:

$$(i) \frac{n^2 + n + 1}{n + 1}, \quad (ii) n^2 \sin n, \quad (iii) \frac{n^{3/4}}{\sqrt{5n - 1}}, \quad (iv) \left(1 + \frac{1}{n}\right)^n.$$

Justify your answers briefly.

turn over/ ...

5. Assume that (a_n) is a sequence such that $a_n \rightarrow 0$. Let (b_n) be a bounded sequence. Prove that $a_n b_n \rightarrow 0$.

Give an example of a single sequence (a_n) such that $a_n \rightarrow 0$ and of appropriate sequences (c_n) to demonstrate that each of the following possibilities can occur:

- (i) $a_n c_n \rightarrow 0$ and (c_n) is unbounded;
- (ii) $a_n c_n \rightarrow \infty$;
- (iii) $(a_n c_n)$ converges to a non-zero limit;
- (iv) $(a_n c_n)$ is bounded and divergent;
- (v) $a_n c_n \rightarrow -\infty$.

6. For each of the following choices of z_n decide whether or not (z_n) converges:

$$(i) \left(\frac{1}{1+i} \right)^n, \quad (ii) \frac{(1-i)n}{n+i}, \quad (iii) (-1)^n \frac{n+i}{n}.$$

Give brief justifications for your answers.

7. [Later parts may be treated as optional] Let c be a complex number. The complex numbers $z_n(c)$ are defined recursively by

$$z_1(c) = c, \quad z_{n+1}(c) = (z_n(c))^2 + c \text{ for } n \geq 1.$$

The *Mandelbrot set* is defined by

$$M = \{ c \in \mathbb{C} \mid \text{the sequence } (z_n(c)) \text{ is bounded} \}.$$

- (a) Show that each of $-2, -1, 0, i$ lies in M but that $1 \notin M$.
- (b) Show that if $c \in M$ then $\bar{c} \in M$, where \bar{c} denotes the conjugate of c .
- (c) Show that if $|c| \leq 1/4$ then $|z_n(c)| < 1/2$ for all n . (So if $|c| \leq 1/4$ then $c \in M$.)
- (d) Show that if $|c| = 2 + \varepsilon$ where $\varepsilon > 0$, then $|z_n(c)| \geq 2 + a_n \varepsilon$ for $n \geq 1$ where $a_n = (4^n + 2)/6$. Deduce that the Mandelbrot set lies entirely within the disc $|z| \leq 2$.

Points to ponder

- A. Let

$$a_n = 1 + \frac{2^{n/1000} - 1}{n^2} \quad \text{and} \quad b_n = \frac{n^2 \cos n}{2^{n/1000} - 1}.$$

Use a calculator to estimate a_n and b_n for $n = 1, 2, 5, 10, 100, 1000$. What do you think is the behaviour of a_n and b_n as n tends to infinity? What confidence do you have in your answers?

- B. Why can you not analyse the limiting behaviour of $(n^{1/n})$ by using the first principles argument applicable to (n^α) , where α is a constant?