

## ANALYSIS I: Problem sheet 4

## Subsequences, Algebra of Limits, Monotonic Sequences

1. (a) Let the sequence  $(a_n)$  be defined by

$$a_n = \left( \frac{n^2 - 1}{n^2 + 1} \right) \cos(2\pi n/3).$$

By considering suitable subsequences prove that  $(a_n)$  diverges.

- (b) Consider the sequence  $(\cos n)$ . Show that, for a suitable positive constant  $K$ , there exist subsequences  $(b_r)$  and  $(c_s)$  of  $(\cos n)$  with  $b_r > K$  for all  $r$  and  $c_s < -K$  for all  $s$ . Deduce that  $(\cos n)$  diverges.
2. (a) Let  $(a_n)$  be a sequence such that the subsequences  $(a_{2n})$  and  $(a_{2n+1})$  both converge to a real number  $L$ . Show that  $(a_n)$  also converges to  $L$ .
- (b) Let  $(b_n)$  be a sequence such that each of the subsequences  $(b_{2n})$ ,  $(b_{2n+1})$ ,  $(b_{3n})$  converges. Need  $(b_n)$  converge? Either provide a proof or a counterexample.
- (c) Let  $(c_n)$  be a sequence such that the subsequence  $(c_{kn})$  converges for each  $k = 2, 3, 4, \dots$ . Need  $(c_n)$  converge? Provide either a proof or a counterexample.
3. For which of the following choices of  $a_n$  does the sequence  $(a_n)$  converge? Justify your answers, and find the value of the limit when it exists.

$$(i) \frac{n^2}{n!}; \quad (ii) \frac{2^n n^2 + 3^n}{3^n(n+1) + n^7}; \quad (iii) \frac{(n!)^2}{(2n)!}; \quad (iv) \frac{n^4 + n^3 \sin n + 1}{5n^4 - n \log n}.$$

[You may freely make use of standard limits and inequalities, sandwiching and AOL methodology, as appropriate.]

4. (a) Let  $(a_n)$  be a real sequence. Prove from the limit definition that  $a_n \geq 0$  and  $a_n \rightarrow L$  implies  $L \geq 0$  and prove further that  $\sqrt{a_n} \rightarrow \sqrt{L}$ .
- (b) Let  $(a_n)$ ,  $(b_n)$  and  $(c_n)$  be sequences of real numbers converging to  $L_1$ ,  $L_2$ ,  $L_3$ , respectively. Let  $d_n = \max\{a_n, b_n, c_n\}$ . Assuming any standard AOL results that you require, prove that  $d_n \rightarrow \max\{L_1, L_2, L_3\}$ .

5. Let  $r > 0$ . Let  $a_n = r^n/n!$ .

- (a) By considering  $a_{n+1}/a_n$  show that the tail  $(a_n)_{n \geq N}$  is monotonic decreasing if  $N$  is sufficiently large. [You should specify a suitable value of  $N$ .]
- (b) Show that  $(a_n)$  converges to a limit  $L$  and find the value of  $L$ .

6. The real sequence  $(a_n)$  is defined by

$$a_1 = c, \quad (\alpha + \beta)a_{n+1} = a_n^2 + \alpha\beta,$$

where  $0 < \alpha < \beta$  and  $c > \alpha$ .

- (a) Prove that if  $(a_n)$  converges to a limit  $L$  then necessarily  $L = \alpha$  or  $L = \beta$ .
- (b) Prove that  $a_{n+1} - \gamma$  and  $a_n - \gamma$  have the same sign, where  $\gamma$  denotes either  $\alpha$  or  $\beta$ .

turn over/ ...

- (c) Prove that, if  $c < \beta$  then  $(a_n)$  converges monotonically to  $\alpha$ . Discuss the limiting behaviour of  $(a_n)$  when  $c \geq \beta$ .
- (d) Prove that, if  $\alpha < c < \beta$ ,

$$|a_n - \alpha| \leq \left( \frac{\alpha + c}{\alpha + \beta} \right)^{(n-1)} (c - \alpha).$$

7. [Optional, to provide additional practice with sequences defined by recurrence relations] Let  $(a_n)$  be the sequence of real numbers given by

$$a_1 = a, \quad a_{n+1} = \frac{2}{a_n + 1} \quad (n \geq 1).$$

- (a) Assume  $0 < a < 1$ . Prove that the subsequences  $(a_{2n})$  and  $(a_{2n+1})$  are monotonic, one increasing and the other decreasing. Prove that each of these subsequences converges, and find their limits. Deduce that  $(a_n)$  converges.
- (b) What happens if  $a > 1$ ?

### Points to ponder

- A. [Infinite limits and results of AOL type] Let  $(a_n)$  and  $(b_n)$  be real sequences and let  $c_n = a_n + b_n$  and  $d_n = a_n - b_n$ . Consider the scenarios
- (a)  $a_n \rightarrow L \in \mathbb{R}$  and  $b_n \rightarrow \infty$ ;
- (b)  $a_n \rightarrow \infty$  and  $b_n \rightarrow M \in \mathbb{R}$ ;
- (c)  $a_n \rightarrow \infty$  and  $b_n \rightarrow \infty$ ;

What are the possible limiting behaviours of  $(c_n)$  and  $(d_n)$  in each case? Find examples to illustrate the possibilities and proofs of any assertions which hold in general.

- B. Let  $x \in \mathbb{R}$ . Let

$$a_{m,n} = \cos\left(\frac{m}{n}\pi x\right).$$

What can you say about the iterated limit

$$\lim_{m \rightarrow \infty} \left( \lim_{n \rightarrow \infty} a_{m,n} \right)?$$

Now consider

$$\lim_{n \rightarrow \infty} \left( \lim_{m \rightarrow \infty} a_{m,n} \right).$$

Does this iterated limit always exist? Exist for some values of  $x$  but not others? Never exist?

What conclusions do you draw?