Perturbation Methods: Problem Sheet 4

Q1(a) $\ddot{x} + \varepsilon \dot{x} + x = 0$

$x = x(t, T)$, $t = \frac{T}{\varepsilon} \Rightarrow \frac{d}{dt} = \frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial T}$

$\Rightarrow x_{tt} + 2\varepsilon x_{tT} + \varepsilon^2 x_{TT} + \varepsilon(x_t + \varepsilon x_T) + x = 0$

$x \sim x_0(t, T) + \varepsilon x_1(t, T) + \dots$ as $\varepsilon \to 0 \Rightarrow$

$O(\varepsilon^0)$: $x_{0tt} + x_0 = 0 \Rightarrow \underline{\underline{x_0 = \frac{1}{2}\left(A(T)e^{it} + \bar{A}(T)e^{-it}\right)}}$

$$\begin{aligned} O(\varepsilon^1): \quad x_{1tt} + x_1 &= -2x_{0tT} - x_{0t} \\ &= -\left(iA_T e^{it} - i\bar{A}e^{-it}\right) - \tfrac{1}{2}\left(iAe^{it} - i\bar{A}e^{-it}\right) \\ &= -i\left(A_T + \tfrac{1}{2}A\right)e^{it} + \text{c.c.} \end{aligned}$$

Can suppress secular terms $e^{\pm it}$ only if $\underline{\underline{A_T + \frac{1}{2}A = 0}}$

$A = Re^{i\Theta} \Rightarrow R_T + iR\Theta_T + \frac{1}{2}R = 0$

$\Rightarrow \Theta_T = 0, \quad R_T = -\frac{1}{2}R$

$\Rightarrow \Theta = \Theta(0), \quad R = R(0)e^{-T/2} \qquad (\Theta(0), R(0) \in \mathbb{R})$

$\Rightarrow \underline{\underline{x_0 = R\cos(t + \Theta) = R(0)e^{-T/2}\cos(t + \Theta(0))}}$

Exact solution is

$$x = r_0 e^{-\varepsilon t/2}\cos\left[\left(1 - \frac{\varepsilon^2}{4}\right)^{1/2} t + \theta_0\right] \qquad (r_0, \theta_0 \in \mathbb{R})$$

$$\sim r_0 e^{-T/2}\cos\left[t + \theta_0 - \frac{\varepsilon^2 t}{8}\right] \qquad \text{as } \varepsilon \to 0$$

$\Rightarrow x - x_0 = O(\varepsilon)$ for $t = O(1/\varepsilon)$ $\square$

(b) $\ddot{x} + x = \varepsilon x^3 \Rightarrow$ same as (a) until

$$O(\varepsilon^1): \quad x_{1tt} + x_1 = -2x_{0tT} - x_0^3$$
$$= -(iA_T e^{it} - i\bar{A}_T e^{-it}) - \tfrac{1}{8}(Ae^{it} + \bar{A}e^{-it})^3$$
$$= \left[-iA_T + \tfrac{3}{8}A^2\bar{A}\right]e^{it} + \text{c.c.} + \text{non-secular}.$$

Can suppress secular terms $e^{\pm it}$ only if $\underline{\underline{iA_T = \frac{3}{8}A^2\bar{A}}}$

$$A = Re^{i\Theta} \Rightarrow \quad i(R_T + iR\Theta_T) = \tfrac{3}{8}R^3$$

$$\Rightarrow \quad R_T = 0, \; R\Theta_T = -\tfrac{3}{8}R^3$$

$$\Rightarrow \quad R = R(0), \; \Theta = \Theta(0) - \tfrac{3}{8}R(0)^2 T$$

$$\Rightarrow \quad A(T) = R(0)\, e^{i\Theta(0) - \frac{3i}{8}R(0)^2 T}$$

$$\Rightarrow \quad \underline{\underline{A(T) = A(0)\, e^{-\frac{3i}{8}|A(0)|^2 T}}}$$

(c) $\ddot{x} + \varepsilon(x^2 - \lambda)\dot{x} + x = 0 \Rightarrow$ same as (a) until

$$O(\varepsilon^1): \quad x_{1tt} + x_1 = -2x_{0tT} - (x_0^2 - \lambda)x_{0t}$$
$$= -(iA_T e^{it} - i\bar{A}_T e^{-it})$$
$$- \left(\tfrac{1}{4}(Ae^{it} + \bar{A}e^{-it})^2 - \lambda\right)(iAe^{it} - i\bar{A}e^{-it})\tfrac{1}{2}$$
$$= \left[-iA_T - \tfrac{1}{4}A^2\left(\tfrac{-i}{2}\bar{A}\right) - \left(\tfrac{2}{4}A\bar{A} - \lambda\right)\tfrac{iA}{2}\right]e^{it}$$
$$+ \text{c.c.} + \text{non-secular}$$

Can suppress secular terms only if $-2iA_T + \frac{i}{4}A^2\bar{A} - i\left(\frac{1}{2}A\bar{A} - \lambda\right)A = 0$

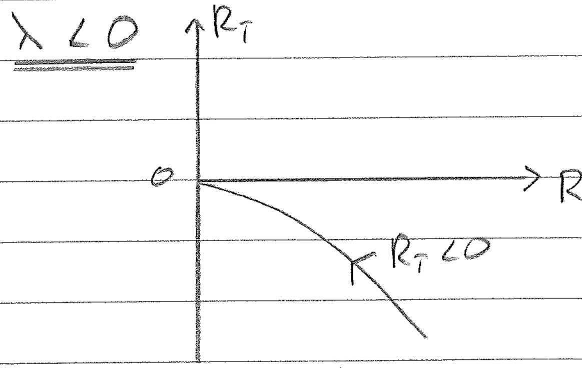
$$\Rightarrow \underline{\underline{2A_T = \left(\lambda - \frac{|A|^2}{4}\right)A}}$$

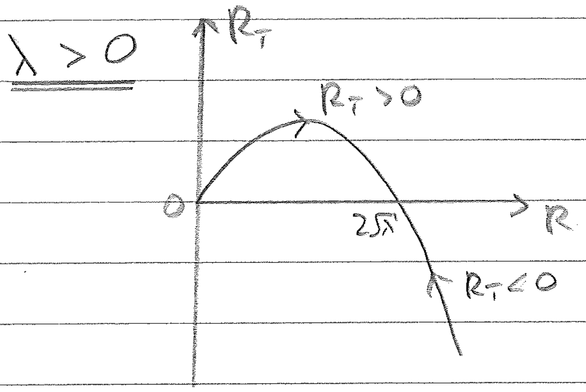
$$A = Re^{i\Theta} \Rightarrow \quad 2(R_T + iR\Theta_T) = \left(\lambda - \frac{R^2}{4}\right)R$$

$\Rightarrow \Theta_T = 0, \quad 2R_T = \left(\lambda - \frac{R^2}{4}\right)R$

$\Rightarrow \Theta(T) = \Theta(0), \quad R(T) \rightarrow \begin{cases} 0, & \lambda < 0 \\ 2\sqrt{\lambda}, & \lambda > 0 \end{cases}$

because $R_T \gtrless 0 \Leftrightarrow \lambda - \frac{R^2}{4} \gtrless 0$ for $R > 0$:





Thus, for $\lambda < 0$, solution tends to only steady solution $R = 0$, while for $\lambda > 0$, solution tends to a periodic orbit with period 2π and amplitude $2\sqrt{\lambda}$ at leading order ($\because x_0 = R(T)\cos(t + \Theta(0))$). This is called a Hopf bifurcation.

$\square$

Q2(a) $\ddot{x} + (1+\varepsilon)x = \cos t.$

$x = x(t, T), \quad t = \frac{T}{\varepsilon} \Rightarrow \frac{d}{dt} = \frac{\partial}{\partial t} + \varepsilon\frac{\partial}{\partial T}$

$\Rightarrow x_{tt} + 2\varepsilon x_{tT} + \varepsilon^2 x_{TT} + (1+\varepsilon)x = \frac{1}{2}(e^{it} + e^{-it})$

Try $x \sim x_0(t,T) + \varepsilon x_1(t,T) + \ldots$ as $\varepsilon \to 0 \Rightarrow$

$O(\varepsilon^0)$ $\quad x_{0tt} + x_0 = \frac{1}{2}(e^{it} + e^{-it})$

$\Rightarrow$ cannot suppress secular $e^{\pm it}$ terms.

Instead expand $x \sim \frac{x_0(t,T)}{\varepsilon} + x_1(t,T) + \ldots$ as $\varepsilon \to 0 \Rightarrow$

$O(\varepsilon^{-1})$: $x_{0tt} + x_0 = 0 \Rightarrow \underline{\underline{x_0 = \frac{1}{2}\left(A(T)e^{it} + \bar{A}(T)e^{-it}\right)}}$

$$O(\varepsilon^0): \quad x_{1tt} + x_1 = -2x_{0tT} - x_0 + \tfrac{1}{2}(e^{it} + e^{-it})$$
$$= -(iA_T e^{it} - i\bar{A}_T e^{-it}) - \tfrac{1}{2}(Ae^{it} + \bar{A}e^{-it}) + \tfrac{1}{2}(e^{it} + e^{-it})$$
$$= [-iA_T - \tfrac{1}{2}A + \tfrac{1}{2}]e^{it} + \text{c.c.}$$

Can suppress secular terms only if $\underline{\underline{-iA_T = \frac{1}{2}(A-1)}}$

$$A = 1 + Re^{i\Theta} \Rightarrow -i(R_T + iR\Theta_T) = \tfrac{1}{2}R$$

$$\Rightarrow \quad R_T = 0, \quad R\Theta_T = \tfrac{1}{2}R$$

$$\Rightarrow \quad R = R(0), \quad \Theta = \Theta(0) + \tfrac{T}{2}$$

$$\Rightarrow \quad A = 1 + R(0)e^{i\Theta(0) + iT/2}$$

$$\Rightarrow \quad x_0 = |A(T)| \cos(t + \arg(A(T)))$$

$\underline{\underline{\text{is periodic (with period } 2\pi) \text{ iff } A(0) = 1}}$

(b) $\ddot{x} + (1+\varepsilon)x + \gamma\varepsilon^3 x^3 = \cos t \Rightarrow$ same as (a) until

$$O(\varepsilon^0): \quad x_{1tt} + x_1 = -2x_{0tT} - x_0 + \tfrac{1}{2}(e^{it} + e^{-it}) - \gamma x_0^3$$
$$= [-iA_T - \tfrac{1}{2}A + \tfrac{1}{2} - \tfrac{3}{8}\gamma A^2\bar{A}) + \text{c.c.} + \text{non-secular}$$

Can suppress secular terms only if $-iA_T - \frac{1}{2}A + \frac{1}{2} - \frac{3}{8}\gamma A^2\bar{A} = 0$

$$A = \alpha(T) + i\beta(T) \Rightarrow -i(\alpha_T + i\beta_T) - \tfrac{1}{2}(\alpha + i\beta) + \tfrac{1}{2} - \tfrac{3}{8}\gamma(\alpha^2 + \beta^2)(\alpha + i\beta) = 0$$

$$\Rightarrow \quad \beta_T + \tfrac{1}{2}(1-\alpha) - \tfrac{3}{8}\gamma(\alpha^2+\beta^2)\alpha = 0$$
$$-\alpha_T - \tfrac{1}{2}\beta - \tfrac{3}{8}\gamma(\alpha^2+\beta^2)\beta = 0$$

Hence, $x_0 = \alpha(T)\cos t - \beta(T)\sin t$ periodic iff $\alpha, \beta = $ constant

$\underset{(\gamma > 0)}{\Leftrightarrow}$ $\alpha + \frac{3}{4}\gamma\alpha^3 = 1$, $\beta = 0$ at $t = 0 \Leftrightarrow \underline{\underline{A(0) \in \mathbb{R},\ A(0) + \frac{3}{4}\gamma A(0)^3 = 1}}$

Q3 $\frac{d}{dx}\left(D\left(x, \frac{x}{\varepsilon}\right)\frac{du}{dx}\right) = f\left(x, \frac{x}{\varepsilon}\right)$

$u = u(x, X)$, $x = \varepsilon X \Rightarrow \frac{d}{dx} = \frac{1}{\varepsilon}\frac{\partial}{\partial X} + \frac{\partial}{\partial x}$

$$\Rightarrow \left(\frac{\partial}{\partial X} + \varepsilon\frac{\partial}{\partial x}\right)\left(D(x,X)\left(\frac{\partial u}{\partial X} + \varepsilon\frac{\partial u}{\partial x}\right)\right) = \varepsilon^2 f(x,X)$$

$u \sim u_0(x,X) + \varepsilon u_1(x,X) + \varepsilon^2 u_2(x,X) + \ldots$ as $\varepsilon \to 0$ gives the following problems at $O(1)$, $O(\varepsilon)$ and $O(\varepsilon^2)$.

$O(1)$: $\quad \underline{\underline{(D u_{0X})_X = 0}}$ (†)

$O(\varepsilon)$: $\quad \underline{\underline{(D(u_{1X} + u_{0x}))_X + (D u_{0X})_x = 0}}$ (††)

$O(\varepsilon^2)$: $\quad \underline{\underline{(D(u_{2X} + u_{1x}))_X + (D(u_{1X} + u_{0x}))_x = f}}$ (†††)

(†) $\Rightarrow \quad D u_{0X} = a_1(x) \quad$ (a_1 arb.)

$\Rightarrow \quad u_0 = a_0(x) + a_1(x)\int_0^X \frac{ds}{D(x,s)} \quad$ (a_0 arb.)

u_0 periodic in X with period 1

$\Rightarrow \quad a_0(x) = u_0(x,0) = u_0(x,1) = a_0(x) + a_1(x)\int_0^1 \frac{ds}{D(x,s)}$

$\Rightarrow \quad a_1 = 0 \quad \because \int_0^1 \frac{ds}{D(x,s)} > 0$

$\Rightarrow \quad \underline{\underline{u_0 = a_0(x) = u_0(x), \text{ say.}}}$

(††) $\Rightarrow \quad D(u_{1X} + u_{0x}) = b_1(x) \quad$ (b_1 arb.)

$\Rightarrow \quad u_1 = b_0(x) - u_{0x}X + b_1(x)\int_0^X \frac{ds}{D(x,s)} \quad$ (b_0 arb.)

u_1 periodic in X with period 1

$\Rightarrow \quad b_0(x) = u_1(x,0) = u_1(x,1) = b_0(x) - u_{0x} + b_1(x)\int_0^1 \frac{ds}{D(x,s)}$

$$\Rightarrow \quad b_1(x) = \hat{D}(x)\, u_{0x} \quad \text{where } \hat{D}(x) := \left(\int_0^1 \frac{ds}{D(x,s)} \right)^{-1},$$

i.e. $\hat{D}$ is the harmonic average of D over one period.

$$(\#) \quad \Rightarrow \quad \big(D(u_{2X} + u_{1x})\big)_X = f - b_{1x}$$

$$\Rightarrow \quad D(u_{2X} + u_{1x}) = c_1(x) + \int_0^X f(x,s)\,ds - b_{1x} X$$

$(c_1 \text{ arb.})$

u_1, u_2 periodic in X with period 1

$$\Rightarrow \quad u_{2X},\ u_{1x}(x,X) = \lim_{h \to 0} \frac{u_1(x+h,X) - u_1(x,X)}{h}$$ periodic in X with period 1

$$\Rightarrow \quad c_1(x) = D(u_{2X} + u_{1x})\big|_{X=0}$$

$$= D(u_{2X} + u_{1x})\big|_{X=1}$$

$$= c_1(x) + \int_0^1 f(x,s)\,ds - b_{1x}$$

$$\Rightarrow \quad b_{1x} = \int_0^1 f(x,s)\,ds$$

Hence, homogenized equation for $u_0(x)$ is

$$\frac{d}{dx}\left(\hat{D}(x) \frac{du_0}{dx} \right) = \hat{f}(x),$$

where

$$\hat{D}(x) = \left(\int_0^1 \frac{ds}{D(x,s)} \right)^{-1}$$

$$\hat{f}(x) = \int_0^1 f(x,s)\,ds$$

Note different averages for the diffusivity $\hat{D}(x)$ and for the capacity $\hat{f}(x)$.

Q4 Let $y = A(x) e^{iu(x)/\varepsilon}$

$$\Rightarrow \quad y' = \left(\frac{iAu'}{\varepsilon} + A' \right) e^{iu/\varepsilon}$$

$$y'' = \left(-\frac{A(u')^2}{\varepsilon^2} + \frac{2iA'u'}{\varepsilon} + \frac{iAu''}{\varepsilon} + A'' \right) e^{iu/\varepsilon}$$

(a) $\varepsilon^2 y'' + xy = 0$ for $x > 0$

$$\Rightarrow \quad -A(u')^2 + 2i\varepsilon A'u' + i\varepsilon Au'' + \varepsilon^2 A'' + xA = 0$$

$A \sim A_0(x) + \varepsilon A_1(x) + \dots$ as $\varepsilon \to 0 \Rightarrow$

$O(\varepsilon^0)$: $-A_0(u')^2 + xA_0 = 0 \underset{(A_0 \neq 0)}{\Rightarrow} u' = \pm x^{1/2}$, $u = \pm \frac{2}{3} x^{3/2}$ (wlog)

$O(\varepsilon^1)$: $-A_1(u')^2 + 2iA_0'u' + iA_0u'' + xA_1 = 0$

$$\Rightarrow \quad 2A_0' x^{1/2} + A_0 \tfrac{1}{2} x^{-1/2} = 0$$

$$\Rightarrow \quad \frac{A_0'}{A_0} = -\frac{1}{4x}$$

$$\Rightarrow \quad \ln|A_0| = c_1 - \frac{1}{4}\ln x \quad (c_1 \in \mathbb{R})$$

$$\Rightarrow \quad A_0 = \frac{c_2}{x^{1/4}} \quad (|c_2| = e^{c_1})$$

Hence, $y_1 \sim \dfrac{c_2^+}{x^{1/4}} e^{\frac{2ix^{3/2}}{3\varepsilon}}$, $y_2 \sim \dfrac{c_2^-}{x^{1/4}} e^{-\frac{2ix^{3/2}}{3\varepsilon}}$ as $\varepsilon \to 0^+$

$(c_2^{\pm} \in \mathbb{R})$

(b) $\varepsilon^2 y'' - xy = 0$ for $x > 0$.

Obtain similarly $u = \pm \dfrac{2ix^{3/2}}{3}$, $A_0 = \dfrac{c_2^{\pm}}{x^{1/4}}$ $\quad (c_2^{\pm} \in \mathbb{R})$

$$\Rightarrow \quad y_1 \sim \frac{c_2^+}{x^{1/4}} e^{-\frac{2x^{3/2}}{3\varepsilon}}, \quad y_2 \sim \frac{c_2^-}{x^{1/4}} e^{+\frac{2x^{3/2}}{3\varepsilon}} \quad \text{as } \varepsilon \to 0^+$$

Valid for $u = O(1)$ as $\varepsilon \to 0$ $\therefore$ lose validity when $x = O(\varepsilon^{2/3})$

Q5 $\varepsilon y'' + y' + xy = 0$ for $0 < x < 1$, with $y(0) = 0$, $y(1) = 1$.

(a) $y = e^{S(x)/\varepsilon} \Rightarrow y' = \frac{S'}{\varepsilon} e^{S/\varepsilon} \Rightarrow y'' = \left[\frac{(S')^2}{\varepsilon^2} + \frac{S''}{\varepsilon} \right] e^{S/\varepsilon}$

ODE $\Rightarrow \quad (S')^2 + S' + \varepsilon(S'' + x) = 0$

$S \sim S_0(x) + \varepsilon S_1(x) + \dots$ as $\varepsilon \to 0^+ \Rightarrow$

$O(\varepsilon^0)$: $(S_0')^2 + S_0' = 0 \Rightarrow S_0' = 0, -1 \Rightarrow S_0 = A_1, B_1 - x$

$(A_1, B_1 \in \mathbb{R})$

$O(\varepsilon^1)$: $2S_0'S_1' + S_1' + S_0'' + x = 0$

$S_0 = A_1 \Rightarrow S_1' = -x \Rightarrow S_1 = A_2 - \frac{1}{2}x^2 \quad (A_2 \in \mathbb{R})$
$S_0 = B_1 - x \Rightarrow S_1' = x \Rightarrow S_1 = B_2 + \frac{1}{2}x^2 \quad (B_2 \in \mathbb{R})$

Hence, general solution $y \sim A_3 e^{-\frac{1}{2}x^2} + B_3 e^{-x/\varepsilon + \frac{1}{2}x^2}$, where $A_3, B_3 \in \mathbb{R}$ and we have absorbed $e^{A_1/\varepsilon + A_2}$ into A_3 and $e^{B_1/\varepsilon + B_2}$ into B_3.

$y(0) = 0 \Rightarrow A_3 \sim -B_3$
$y(1) = 1 \Rightarrow A_3 e^{-1/2} + B_3 e^{-1/\varepsilon + 1/2} \sim 1$

Thus, $A_3 \sim -B_3 \sim \dfrac{1}{e^{-1/2} - e^{-1/\varepsilon + 1/2}} = \dfrac{e^{1/2}}{1 - e^{1 - 1/\varepsilon}}$

giving $$y \sim \frac{e^{(1-x^2)/2} - e^{-x/\varepsilon + (1+x^2)/2}}{1 - e^{1-1/\varepsilon}} \quad \text{as } \varepsilon \to 0^+$$

(b) $x = 1 + \varepsilon X$, $y = Y(X) \Rightarrow \frac{d^2Y}{dX^2} + \frac{dY}{dX} + \varepsilon(1 + \varepsilon X)Y = 0$
$\Rightarrow Y \sim C_1 + C_2 e^{-X} \quad (C_1, C_2 \in \mathbb{R})$ as $\varepsilon \to 0^+$

Matching with outer ($1 - x = O(1)$) requires $Y(-\infty)$ to be finite
$\Rightarrow C_2 = 0 \Rightarrow$ no BL at $x = 1$ at leading order.

<u>Outer</u>: $y \sim y_0(x) + \varepsilon y_1(x) + \ldots$ as $\varepsilon \to 0^+$ with $x = O(1)$ $\Rightarrow$

$O(\varepsilon^0)$: $y_0' + x y_0 = 0$ for $0 < x < 1$, with $y_0(1) = 1$ ($\because$ no BL at $x = 1$)

$\Rightarrow \dfrac{y_0'}{y_0} = -x \Rightarrow \ln|y_0| = D_1 - \dfrac{x^2}{2} \Rightarrow y_0 = D_2 e^{-x^2/2}$ ($D_1, D_2 \in \mathbb{R}$, $|D_2| = e^{D_1}$)

$y_0(1) = 1 \Rightarrow 1 = D_2 e^{-1/2} \Rightarrow \underline{\underline{y_0(x) = e^{(1-x^2)/2}}}$

<u>Inner</u>: $x = \varepsilon X$, $y = Y(X) \Rightarrow \dfrac{d^2Y}{dX^2} + \dfrac{dY}{dX} + \varepsilon^2 X Y = 0$ (balancing 1st and 2nd terms)

$Y \sim Y_0(X) + \varepsilon Y_1(X) + \ldots$ as $\varepsilon \to 0^+$ with $X = O(1)$ $\Rightarrow$

$O(\varepsilon^0)$: $\dfrac{d^2Y_0}{dX^2} + \dfrac{dY_0}{dX} = 0$ for $X > 0$, with $Y_0(0) = 0$ (BC)

$\Rightarrow \underline{\underline{Y_0 = E_1(1 - e^{-X}) \quad (E_1 \in \mathbb{R})}}$

<u>Matching</u>:

$(1.t.o.) = e^{(1-x^2)/2}$

$\Rightarrow (1.t.o.)$ in inner variables $= e^{(1-\varepsilon^2 X^2)/2} \sim e^{1/2}$

$\Rightarrow (1.t.i.)(1.t.o.) = e^{1/2}$

$(1.t.i.) = E_1(1 - e^{-X})$

$\Rightarrow (1.t.i.)$ in outer variables $= E_1(1 - e^{-x/\varepsilon}) \sim E_1$

$\Rightarrow (1.t.o.)(1.t.i.) = E_1$

$(1.t.i.)(1.t.o.) = (1.t.o.)(1.t.i.) \Rightarrow \underline{\underline{E_1 = e^{1/2}}}$

<u>Composite expansion</u>: Additive composite expansion given by

$$y \sim y_0(x) + Y_0(x/\varepsilon) - (1.t.i.)(1.t.o.)$$

$$\underline{\underline{= e^{(1-x^2)/2} - e^{1/2 - x/\varepsilon} \quad \text{as } \varepsilon \to 0^+}}$$

(because $(1.t.i.)(1.t.o.)$ counted twice in $y_0(x) + Y_0(x/\varepsilon)$).

Q6 $\varepsilon^2 y'' + (1-x)y = 0$ for $x > 0$, with $y(0) = 1$, $y(\infty) = 0$.

(a) Let $x = 1 + \varepsilon^{2/3} X$, $y = Y(X) \Rightarrow \dfrac{d^2 Y}{dX^2} - XY = 0$ for $X > -\varepsilon^{-2/3}$

$\Rightarrow Y(X) = a\,\mathrm{Ai}(X) + b\,\mathrm{Bi}(X) \qquad (a, b \in \mathbb{R})$

BCs become $Y(-\varepsilon^{-2/3}) = 1$, $Y(\infty) = 0$.

As $X \to \infty$, $\mathrm{Ai}(X) \sim \dfrac{1}{2\sqrt{\pi}X^{1/4}} e^{-\frac{2}{3}X^{3/2}}$, $\mathrm{Bi}(X) \sim \dfrac{1}{\sqrt{\pi}X^{1/4}} e^{\frac{2}{3}X^{3/2}}$

$Y(\infty) = 0 \Rightarrow b = 0$

$Y(-\varepsilon^{-2/3}) = 1 \Rightarrow a\,\mathrm{Ai}(-\varepsilon^{-2/3}) = 1$

Hence, exact solution is $y(x) = Y(X) = \dfrac{\mathrm{Ai}(X)}{\mathrm{Ai}(-\varepsilon^{-2/3})} = \dfrac{\mathrm{Ai}(\varepsilon^{-2/3}(x-1))}{\mathrm{Ai}(-\varepsilon^{-2/3})}$

(b) $y = A(x)e^{i\phi(x)/\varepsilon} \Rightarrow y' = \left(\dfrac{iA\phi'}{\varepsilon} + A'\right)e^{i\phi/\varepsilon}$

$\Rightarrow y'' = \left(-\dfrac{A(\phi')^2}{\varepsilon^2} + \dfrac{2iA'\phi'}{\varepsilon} + \dfrac{iA\phi''}{\varepsilon} + A''\right)e^{i\phi/\varepsilon}$

ODE $\Rightarrow -A(\phi')^2 + \varepsilon(2iA'\phi' + iA\phi'') + \varepsilon^2 A'' + (1-x)A = 0$

$A \sim A_0(x) + \varepsilon A_1(x) + \dots$ as $\varepsilon \to 0 \Rightarrow$

$O(\varepsilon^0)$: $-A_0(\phi')^2 + (1-x)A_0 \Rightarrow \phi' = \pm(1-x)^{1/2} \Rightarrow \phi = \pm\frac{2}{3}(1-x)^{\frac{3}{2}} + \text{const}$

$O(\varepsilon^1)$: $-A_1(\phi')^2 + 2iA_0'\phi' + A_0\phi'' + (1-x)A_1 = 0 \Rightarrow (A_0^2\phi')' = 0$

$\Rightarrow A_0^2 = \dfrac{\text{constant}}{\phi'} \Rightarrow A_0 = \dfrac{\text{constant}}{(1-x)^{1/4}}$

Hence, character of solution changes depending on whether $x > 1$ or $x < 1$

RH outer $x > 1$

$y(\infty) = 0 \Rightarrow$ need to eliminate growing solution, giving

$$y \sim \frac{C_1}{(x-1)^{1/4}} \exp\left(-\frac{2}{3\varepsilon}(x-1)^{3/2}\right) \text{ as } \varepsilon \to 0^+ \text{ with } x > 1,\ x-1 = \operatorname{ord}(1)$$

where $C_1 \in \mathbb{R}$

LH outer $0 < x < 1$

Now both $\phi = \pm\frac{2}{3}(1-x)^{3/2}$ are admissible, giving

$$y \sim \frac{C_2}{(1-x)^{1/4}} \sin\left(\frac{2}{3\varepsilon}(1-x)^{3/2} + \alpha_1\right) \text{ as } \varepsilon \to 0^+ \text{ with } 0 < x < 1,\ x = \operatorname{ord}(1)$$

where $C_2, \alpha_1 \in \mathbb{R}$

$$y(0) = 1 \Rightarrow C_2 \Rightarrow y \sim \frac{\operatorname{cosec}\left(\frac{2}{3\varepsilon} + \alpha_1\right)}{(1-x)^{1/4}} \sin\left(\frac{2}{3\varepsilon}(1-x)^{3/2} + \alpha_1\right)$$

Inner region near $x = 1$

Outer solutions unbounded as $x \to 1^\pm$, so seek an inner solution by scaling $x = 1 + \delta(\varepsilon)X$, $y = \delta(\varepsilon)^{-1/4} Y(X)$, giving Airy's equation $\frac{d^2Y}{dX^2} = XY$ provided $\delta(\varepsilon) = \varepsilon^{2/3}$.

General solution is

$$Y(X) = C_3 \operatorname{Ai}(X) + C_4 \operatorname{Bi}(X) \qquad (C_3, C_4 \in \mathbb{R})$$

Matching inner ($X \to \infty$) with RH outer ($x \to 1^+$)

Safer to use intermediate variable $\hat{x}$ to match:

$$x - 1 = \varepsilon^\alpha \hat{x} = \varepsilon^{2/3} X \qquad (0 < \alpha < \tfrac{2}{3}).$$

As $X \to \infty$, $Ai(X) \sim \frac{1}{2\sqrt{\pi}X^{1/4}} e^{-\frac{2}{3}X^{3/2}}$, $Bi(X) \sim \frac{1}{\sqrt{\pi}X^{1/4}} e^{\frac{2}{3}X^{3/2}}$

- $X = \frac{\hat{x}}{\varepsilon^{2/3-\alpha}} \to \infty$ as $\varepsilon \to 0^+$ with $\hat{x} > 0$, $\hat{x} = \text{ord}(1) \Rightarrow$

$$\delta^{-1/4} Y\left(\frac{\hat{x}}{\varepsilon^{2/3-\alpha}}\right) = \frac{C_3}{\varepsilon^{1/6}} Ai\left(\frac{\hat{x}}{\varepsilon^{2/3-\alpha}}\right) + \frac{C_4}{\varepsilon^{1/6}} Bi\left(\frac{\hat{x}}{\varepsilon^{2/3-\alpha}}\right)$$
$$\sim \frac{C_3}{\varepsilon^{1/6}} \frac{1}{2\sqrt{\pi}(\hat{x}/\varepsilon^{2/3-\alpha})^{1/4}} \exp\left(-\frac{2}{3}\left(\frac{\hat{x}}{\varepsilon^{2/3-\alpha}}\right)^{3/2}\right)$$
$$+ \frac{C_4}{\varepsilon^{1/6}} \frac{1}{\sqrt{\pi}(\hat{x}/\varepsilon^{2/3-\alpha})^{1/4}} \exp\left(\frac{2}{3}\left(\frac{\hat{x}}{\varepsilon^{2/3-\alpha}}\right)^{3/2}\right)$$

$y \sim \frac{C_1}{(x-1)^{1/4}} \exp\left(-\frac{2}{3\varepsilon}(x-1)^{3/2}\right)$

- $x = 1 + \varepsilon^{\alpha}\hat{x} \to 1^+$ as $\varepsilon \to 0^+$ with $\hat{x} > 0$, $\hat{x} = \text{ord}(1) \Rightarrow$

$$y(1+\varepsilon^{\alpha}\hat{x}) \sim \frac{C_1}{(\varepsilon^{\alpha}\hat{x})^{1/4}} \exp\left(-\frac{2}{3\varepsilon}(\varepsilon^{\alpha}\hat{x})^{3/2}\right)$$

- Matching $\Rightarrow$ $C_4 = 0$, $C_1 = \frac{C_3}{2\sqrt{\pi}}$

Matching inner ($X \to -\infty$) with LH outer ($x \to 1^-$)

- $X = \frac{\hat{x}}{\varepsilon^{2/3-\alpha}} \to -\infty$ as $\varepsilon \to 0^+$ with $\hat{x} < 0$, $\hat{x} = \text{ord}(1) \Rightarrow$

$$\delta^{-1/4} Y\left(\frac{\hat{x}}{\varepsilon^{2/3-\alpha}}\right) = \frac{C_3}{\varepsilon^{1/6}} Ai\left(\frac{\hat{x}}{\varepsilon^{2/3-\alpha}}\right) \sim \frac{C_3}{\varepsilon^{1/6}} \frac{1}{\sqrt{\pi}(-\hat{x}/\varepsilon^{2/3-\alpha})^{1/4}} \sin\left(\frac{2}{3}\left(\frac{-\hat{x}}{\varepsilon^{2/3-\alpha}}\right)^{3/2} + \frac{\pi}{4}\right)$$

- $x = 1 + \varepsilon^{\alpha}\hat{x} \to 1^-$ as $\varepsilon \to 0^+$ with $\hat{x} < 0$, $\hat{x} = \text{ord}(1) \Rightarrow$

$$y(1+\varepsilon^{\alpha}\hat{x}) \sim \frac{\operatorname{cosec}\left(\frac{2}{3\varepsilon} + \alpha_1\right)}{(-\varepsilon^{\alpha}\hat{x})^{1/4}} \sin\left(\frac{2}{3\varepsilon}(-\varepsilon^{\alpha}\hat{x})^{3/2} + \alpha_1\right)$$

- Matching $\Rightarrow$ $\alpha_1 = \frac{\pi}{4}$ (wlog), $\frac{C_3}{\sqrt{\pi}} = \operatorname{cosec}\left(\frac{2}{3\varepsilon} + \frac{\pi}{4}\right)$

- Hence, $C_1 = \frac{1}{2}\operatorname{cosec}\left(\frac{2}{3\varepsilon} + \frac{\pi}{4}\right)$ and we're done.

- NB: plots show excellent agreement with exact solution for $\varepsilon \lesssim 0.1$