ANALYSIS I: Problem sheet 5

Cauchy Sequences, Series (convergence and absolute convergence), Properties of e, Alternating Series Test.

1. For $n \in \mathbb{N}$ let
$$a_n = \int_1^n \frac{\cos x}{x^2}\,\mathrm{d}x.$$
Prove, for $m \geqslant n \geqslant 1$, that $|a_m - a_n| \leqslant n^{-1}$ and deduce that (a_n) converges. By integration by parts, or otherwise, demonstrate the existence of
$$\lim_{n\to\infty} \int_1^n \frac{\sin x}{x}\,\mathrm{d}x.$$

2. (a) Let p be a non-zero natural number. Prove by considering the partial sums that
$$\sum \frac{1}{k(k+p)}$$
converges. What is $\sum_{k=1}^{\infty} 1/(k(k+p))$?

(b) Evaluate the sum
$$\sum_{k=1}^{\infty} \frac{\cos k}{2^k}.$$

(c) Use the Comparison Test to prove that
$$\sum \frac{2k+1}{(k+1)(k+2)^2}$$
converges.

3. Let (a_n) be a sequence of real or complex numbers and assume that $\sum_{k=1}^{\infty} |a_k|$ converges. Prove, by considering the partial sums
$$s_n = a_1 + \cdots + a_n \qquad \text{and} \qquad S_n = |a_1| + \cdots + |a_n|,$$
that
$$\left|\sum_{k=1}^{\infty} a_k\right| \leqslant \sum_{k=1}^{\infty} |a_k|.$$
[*You may assume the Triangle Law for real or complex numbers and the theorem that* $\sum |a_k|$ *converges implies that* $\sum a_k$ *converges.*]

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4. The number known as e is defined by

$$\mathrm{e} = \lim_{n\to\infty} s_n, \quad \text{where } s_n = \sum_{k=0}^{n} \frac{1}{k!}.$$

(a) Prove that (s_n) is monotonic increasing and bounded above and deduce that the limit defining e exists. [*Hint for getting an upper bound: compare s_n with the sum of a geometric progression.*]

(b) Show that, for $n \geqslant 1$,

$$0 < \mathrm{e} - \sum_{k=0}^{n} \frac{1}{k!} < \frac{1}{n!\,n}.$$

Deduce that e is irrational.

5. There exists a real number L such that

$$\left(1 + \frac{1}{n}\right)^n \to L \quad \text{as } n \to \infty$$

(recall Example 9.3(a)). In fact $L = \mathrm{e}$ [a proof is given in the supplementary notes on *e*]. Assuming this fact, show that

$$\left(1 - \frac{1}{n}\right)^n \to \frac{1}{\mathrm{e}} \quad \text{as } n \to \infty.$$

6. (a) Prove that $\sum(-1)^{k-1}(\sqrt{k+1} - \sqrt{k})$ converges.

(b) Let

$$s_n = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{(-1)^{n+1}}{2n-1}.$$

The Alternating Series Test implies that (s_n) converges to some limit L. By examining the proof of the AST prove that $2/3 < L < 13/15$.

7. Let $\sum a_k$ be a series of real numbers. Which of the following are true and which are false? Provide a proof or counterexample as appropriate.

(a) $k^2 a_k \to 0$ implies $\sum a_k$ converges.

(b) If $\sum a_k$ converges, then $\sum (a_k)^2$ converges.

(c) If $\sum a_k$ converges absolutely, then $\sum (a_k)^2$ converges.

(d) $\sum (a_k)^2$ convergent implies $\sum (a_k)^3$ convergent.

8. [Optional, and quite challenging] For each of the following statements either provide a proof or a counterexample.

(a) For a divergent series $\sum a_k$ of positive terms, $\sum \dfrac{a_k}{1+a_k}$ is also divergent.

(b) Assume $a_k > 0$. Then $\sum a_k$ and $\sum a_k/s_k$ either both converge or both diverge, where $s_k = a_1 + \cdots + a_k$.

Points to ponder

A. Give an example of a real sequence (c_n) for which $|c_{n+1} - c_n| \to 0$ as $n \to \infty$ but for which (c_n) fails to converge. What does the existence of such an example tell you?

B. Let (a_k) be a sequence. Define new sequences (b_k) and (c_k) as follows:

$$\begin{aligned} & b_k = (a_{2k-1} + a_{2k}) && \text{for } k \geqslant 1; \\ c_1 = a_1, \quad & c_k = (a_{2(k-1)} + a_{2k-1}) && \text{for } k \geqslant 2. \end{aligned}$$

Now let $a_k = (-1)^k$ for each $k \geqslant 1$. Calculate

$$\begin{aligned} s_n &:= a_1 + a_2 + \cdots + a_n; \\ t_n &:= b_1 + b_2 + \cdots + b_n; \\ u_n &:= c_1 + c_2 + \cdots + c_n. \end{aligned}$$

What are the limiting behaviours of (s_n), (t_n) and (u_n)? What happens for other choices of (a_k)? What do your answers tell you about infinite sums? Specifically, do such sums always behave in the same way as finite sums do?