ANALYSIS I: Problem sheet 6

Convergence of Series, Tests for Convergence

Throughout you may assume results seen earlier concerning limits of particular sequences—a chance to consolidate your knowledge of useful limits!

1. For the following choices of a_k, use the indicated test to establish whether or not $\sum a_k$ converges.

   (a) $\dfrac{(2k+1)(3k-1)}{(k+1)(k+2)^2}$ (Comparison Test, limit form);

   (b) $\dfrac{1}{k^{1/k}k}$ (Comparison Test, limit form);

   (c) $\dfrac{k!}{k^k}$ (Ratio Test);

   (d) $\binom{2k}{k}^{-1}(4-10^{-23})^k$ (Ratio Test);

   (e) $2^{-k}k$ if k is of the form 2^m and 0 otherwise (Ratio Test).

   Write brief comments on possible alternative approaches, if any, you might have tried on these examples.

2. (a) Use the Integral Test to prove that $\displaystyle\sum_{k\geqslant 3}\frac{1}{k(\log k)^p}$ converges if $p>1$ and diverges if $0<p\leqslant 1$ ($p\in\mathbb{R}$).

   (b) For which positive values of α (α not assumed to be a natural number) does $\displaystyle\sum\frac{k^{-\alpha}}{1+\alpha^{-k}}$ converge?

3. [A miscellany of examples. Standard convergence tests may be assumed where required] For which of the following choices of a_k is $\sum a_k$ convergent? Justify your answers briefly.

   (a) $\dfrac{k+2^k}{2^k k}$; (b) $\dfrac{1}{k}\sin\dfrac{1}{k}$; (c) $\dfrac{\sinh k}{2^k}$;

   (d) $(-1)^{k-1}\dfrac{\log k}{\sqrt{k}}$; (e) $\begin{cases}1/k^2 & \text{if } k \text{ is odd,}\\ -(\log k)/k^2 & \text{if } k \text{ is even;}\end{cases}$ (f) $\left(1-\dfrac{1}{k}\right)^{k^2}$.

turn over/ ...

4. For each of the following power series $\sum c_k x^k$ establish which of the following is true: (i) $\sum |c_k x^k|$ converges for all $x \in \mathbb{R}$; (ii) $\sum |c_k x^k|$ converges only for $x = 0$; (iii) $\sum |c_k x^k|$ converges for $|x| < R$ and diverges for $|x| > R$ for some $R \in \mathbb{R}^{>0}$, which you should determine.

$$\text{(a)} \ \sum k^{2015} x^k; \qquad \text{(b)} \ \sum \frac{x^k}{2^k k^4}; \qquad \text{(c)} \ \sum 2^k x^{k!};$$

$$\text{(d)} \ \sum k^k x^k; \qquad \text{(e)} \ \sum \frac{1}{(4k)!} x^{2k}; \qquad \text{(f)} \ \sum \sin(k) x^k.$$

What can you say in each case about the values of x for which $\sum c_k x^k$ converges?

[*This is a familiarisation exercise on the notion of the radius of convergence of a power series. The intention is that you should make use of your knowledge of convergence tests for series to discover how the given power series behave, not that you should attempt to call on general results from Section 13 of the webnotes or from elsewhere.*]

5. Give either a justification or a counterexample for each of the following statements about real series.

   (a) If $ka_k \to 0$ as $k \to \infty$ then $\sum a_k$ converges.

   (b) If $\lim\limits_{k\to\infty} \dfrac{a_{k+1}}{a_k}$ exists and equals L, where $L > 1$, then $\sum a_k$ diverges.

   (c) If $\sum a_k$ converges and $a_k/b_k \to 1$ then $\sum b_k$ converges.

6. [Optional, or for use for consolidation later)]

   (a) Prove that the series $\sum\limits_{k\geqslant 2} (\log k)^{-k}$ and $\sum\limits_{k\geqslant 2} (\log k)^{-\log k}$ converge.

   (b) Prove that, for any constant $\alpha > 0$, the series $\sum\limits_{k\geqslant 2} (\log k)^{-\alpha}$ diverges.

Points to Ponder

A. A student is doing a homework exercise which asks for a statement of the simple Comparison Test and writes

$$0 \leqslant \sum a_k \leqslant \sum b_k \implies \sum a_k \text{ converges if } \sum b_k \text{ converges.}$$

Is this a valid formulation of the test and if not, why not?

B. Why is it not admissible to establish the convergence or divergence of a geometric series by using the Ratio Test?

C. Consider a real series $\sum (-1)^{k-1} u_k$ with $u_k \geqslant 0$. The Alternating Series Test imposes two further conditions on (u_k) to guarantee that the series will converge. Exhibit two divergent alternating series, one of which fails the first of these conditions and satisfies the second and another which satisfies the first condition and fails to satisfy the second. Check how the standard AST proof would fail on your examples.

What is the moral of this question?