ANALYSIS I: Problem sheet 7

Conditionally Convergent Series; Power Series

1. (a) Prove that
$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots = \frac{3}{2}\log 2.$$

   (b) [Optional, for extra practice] Calculate the value of
$$\sum_{k=1}^{\infty} \frac{1}{k(9k^2-1)}.$$

2. [Consolidation] Find the radius of convergence of the following real power series (assume $k \geqslant 1$ where necessary):
$$\text{(a)} \ \sum (-1)^k k^2 x^k; \quad \text{(b)} \ \sum \frac{(2k-1)(2k-3)\cdots 3\cdot 1}{k!} x^k; \quad \text{(c)} \ \sum (x/k)^k; \quad \text{(d)} \ \sum k^{1/k} x^k.$$

3. (a) Give examples of real power series $\sum c_k x^k$ with the specified radius of convergence R and specified behaviour at $\pm R$:
   (i) $R = 1$ and the power series converges at -1 but diverges at 1;
   (ii) $R = 1$ and the power series converges at 1 and diverges at -1;
   (iii) $R = 2$ and the power series converges at 2 and -2;
   (iv) $R = 2$ and the power series diverges at 2 and -2.

   (b) Let the real power series $\sum a_k x^k$, $\sum b_k x^k$ and $\sum c_k x^k$ have radius of convergence R, S and T, respectively, where $c_k = a_k + b_k$. Obtain a lower bound for T involving R and S. Provide examples to illustrate what possibilities can arise.

4. For which real values of x does $\sum x^k$ converge? Use the Differentiation Theorem for power series to evaluate
$$\text{(i)} \ \sum_{k=1}^{\infty} k x^k; \qquad \text{(ii)} \ \sum_{k=1}^{\infty} k^2 x^k,$$
specifying where the formulae you obtain are valid.

5. (a) Prove that the power series
$$\sum \frac{x^k}{(2k)!} \quad \text{and} \quad \sum \frac{x^k}{(2k+1)!}$$
have infinite radius of convergence.

   (b) Define
$$p(x) = \sum_{k=0}^{\infty} \frac{x^k}{(2k)!} \quad \text{and} \quad q(x) = \sum_{k=0}^{\infty} \frac{x^k}{(2k+1)!}.$$
Use the Differentiation Theorem to compute $p'(x)$ and $q'(x)$ and prove that $2p'(x) = q(x)$ and $p(x) - q(x) = 2xq'(x)$. Hence prove that, for all x,
$$(p(x))^2 = 1 + x(q(x))^2.$$

turn over/...

6. Find the radius of convergence of the power series defining the function J_0, where

$$J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{x}{2}\right)^{2k}.$$

Assuming term-by-term differentiation is allowed within the interval of convergence show that $y = J_0(x)$ is a solution of the equation

$$xy'' + y' + xy = 0.$$

7. [Optional] The *Fibonacci numbers* F_n are defined by $F_0 = 0$, $F_1 = 1$ and $F_{k+2} = F_{k+1} + F_k$ for $k \geqslant 0$. Define

$$F(x) = \sum_{k=0}^{\infty} F_k x^k.$$

Determine the radius of convergence the series defining $F(x)$. By summing the identity

$$F_{k+2}x^k = F_{k+1}x^k + F_k x^k$$

from $k = 0$ to ∞, or otherwise, find $F(x)$ in closed form.

8. [Optional: addition formula for sinh] Show that each of the power series $\sum \frac{x^{2k}}{(2k)!}$ and $\sum \frac{x^{2k+1}}{(2k+1)!}$ converges for all $x \in \mathbb{R}$. Define

$$C(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} \quad \text{and} \quad S(x) = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}.$$

(a) Assuming the Differentiation Theorem for power series, calculate the derivatives of $C(x)$ and $S(x)$.

(b) For fixed $d \in \mathbb{R}$, let

$$f_d(x) = S(d+x)C(d-x) + S(d-x)C(d+x).$$

By considering the derivative of $f_d(x)$, prove that, for all $a, b \in \mathbb{R}$,

$$S(a+b) = S(a)C(b) + S(b)C(a).$$

9. [An optional vacation assignment] In a separate file (Supplement to Problem sheet 7) you will find part of an antique first-year exam paper. Questions 2 and 3 together lead you through a proof of Stirling's approximation to $n!$. Have a go at these questions.

A point to ponder

A. Assume that $\sum a_k z^k$ and $\sum b_k z^k$ are complex power series which are both absolutely convergent for $|z| < R$. Let

$$c_m = \sum_{r=0}^{m} a_r b_{m-r} \quad \text{for } m \geqslant 0.$$

Then it can be proved (but won't be proved in Analysis I) that $\sum c_m z^m$ converges absolutely for $|z| < R$ and that

$$\sum_{m=0}^{\infty} c_m z^m = \left(\sum_{k=0}^{\infty} a_k z^k\right)\left(\sum_{\ell=0}^{\infty} b_\ell z^\ell\right).$$

This is the Multiplication Theorem for power series. It says that the product of two power series can be obtained in the expected way, by collecting together in the product the coefficients of terms in z^m for each m.

It is possible to obtain various formulae involving power series by multiplication: examples are the formula $\mathrm{e}^{a+b} = \mathrm{e}^a \mathrm{e}^b$; the addition formulae for the trigonometric and hyperbolic functions; obtaining a power series expansion of $(1-x)^{-2}$ (see Q. 4 above). Experiment with proving a few of these results by using the Multiplication Theorem.