Linear Algebra I, Sheet 1, MT2018

Systems of linear equations; matrices and their algebra.

1. Use any method you can think of to decide which (if any) of the following systems of linear equations with real coefficients have no solutions, which have a unique solution (in which case, what is it?), which have infinitely many solutions.

$$\text{(a)} \left\{ \begin{array}{rcl} 2x+4y-3z &=& 0 \\ x-4y+3z &=& 0 \\ 3x-5y+2z &=& 1 \end{array} \right. ; \quad \text{(b)} \left\{ \begin{array}{rcl} x+2y+3z &=& 0 \\ 2x+3y+4z &=& 1 \\ 3x+4y+5z &=& 2 \end{array} \right. ; \quad \text{(c)} \left\{ \begin{array}{rcl} x+2y+3z &=& 0 \\ 2x+3y+4z &=& 2 \\ 3x+4y+5z &=& 2 \end{array} \right. .$$

2. Let $A := \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B := \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$, $C := \begin{pmatrix} -1 & 5 \\ 4 & 4 \end{pmatrix}$, $D := \begin{pmatrix} 1 & 4 & -3 \\ 2 & 4 & -2 \end{pmatrix}$, $E := \begin{pmatrix} 1 & 2 \end{pmatrix}$ and $F := \begin{pmatrix} -1 & 5 & -6 \\ 3 & 4 & -1 \end{pmatrix}$. For which pairs $X, Y \in \{A, B, C, D, E, F\}$ is $X - 2Y$ defined? And when it is defined, calculate it.

3. Calculate the following matrix products:

$$\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}; \quad \begin{pmatrix} 0 & 1 \\ 2 & 3 \\ 4 & 6 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}; \quad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}^2 .$$

4. Square matrices A and B of the same size are said to *commute* (with respect to multiplication) if $AB = BA$. Let A be the 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

(a) Show that A commutes with $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ if and only if A is diagonal (that is, $b = c = 0$).

(b) Which 2×2 matrices A commute with $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$?

(c) Use the results of (a) and (b) to find the matrices A that commute with every 2×2 matrix.

5. Check that you know the definition of the *transpose* of a matrix (it's in the notes). Let A and B be $m \times n$ matrices, and let C be an $n \times p$ matrix.

(a) Show that $(A + B)^T = A^T + B^T$ and that $(\lambda A)^T = \lambda A^T$ for scalars λ.

(b) Show that $(AC)^T = C^T A^T$.

(c) Suppose that $m = n$ and A^{-1} exists. Show that A^T is invertible and that $(A^T)^{-1} = (A^{-1})^T$.

6. Let A and B denote square matrices with real entries. For each of the following assertions, find either a proof or a counterexample.

(a) $A^2 - B^2 = (A - B)(A + B)$.

(b) If $AB = 0$ then $A = 0$ or $B = 0$.

(c) If $AB = 0$ then A and B cannot both be invertible.

(d) If A and B are invertible then $A + B$ is invertible.

(e) If $ABA = 0$ and B is invertible then $A^2 = 0$.

[Hint: where the assertions are false there are usually counterexamples of size 2×2.]