Linear Algebra I, Sheet 3, MT2018

Vector spaces and subspaces. Spanning sets. Linear independence. Bases.

1. Show that the set of real sequences (u_n) that satisfy the recurrence relation $u_{n+1} = u_n + u_{n-1}$ (for $n \geqslant 1$) is a real vector space (a subspace of the space of all sequences of real numbers).

Find a basis, and write down the dimension of the vector space.

2. For each of the following vector spaces and each of the specified subsets, determine whether or not the subset is a subspace. That is, in each case, either verify the conditions defining a subspace (or use the subspace test), or show by an example that one of the conditions does not hold.

(a) $V = \mathbb{R}^4$: (i) $\{(a, b, c, d) \in V : a + b = c + d\}$;
(ii) $\{(a, b, c, d) \in V : a + b = 1\}$;
(iii) $\{(a, b, c, d) \in V : a^2 = b^2\}$.

(b) $V = \mathcal{M}_{n\times n}(\mathbb{R})$: (i) the set of upper triangular matrices;
(ii) the set of invertible matrices;
(iii) the set of matrices that are not invertible.

3. Let S be a finite spanning set for a vector space V. Let T be a smallest subset of S that spans V. Show that T is linearly independent, hence a basis of V.

4. (a) Which of the following sets of vectors in $\mathbb{R}^3$ are linearly independent?

(i) $\{(1, 3, 0), (2, -3, 4), (3, 0, 4)\}$, (ii) $\{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$.

(b) Let $V := \mathbb{R}^{\mathbb{R}} = \{f : \mathbb{R} \to \mathbb{R}\}$. Which of the following sets are linearly independent in V?
(i) $\{f, g, h\}$ where $f(x) = 5x^2 + x + 1$, $g(x) = 2x + 3$ and $h(x) = x^2 - 1$.
(ii) $\{p, q, r\}$ where $p(x) = \cos^2(x)$, $q(x) = \cos(2x)$ and $r(x) = 1$.

5. (a) Let u, v, w be linearly independent vectors in a vector space V.
(i) Show that $u + v$, $u - v$, $u - 2v + w$ are also linearly independent.
(ii) Are $u + v - 3w$, $u + 3v - w$, $v + w$ linearly independent?

(b) Let $\{v_1, v_2, \ldots, v_n\}$ be a linearly independent set of n vectors in a vector space V. Prove that each of the following sets is also linearly independent:
(i) $\{c_1v_1, c_2v_2, \ldots, c_nv_n\}$ where $c_i \neq 0$ for $1 \leqslant i \leqslant n$;
(ii) $\{w_1, w_2, \ldots, w_n\}$ where $w_i = v_i + v_1$ for $1 \leqslant i \leqslant n$.

6. (a) Let $V_1 := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 + \cdots + x_n = 0\}$. Show that V_1 is a subspace of $\mathbb{R}^n$ and find a basis for it.

(b) Let $V_2 := \{(x_{ij}) \in \mathcal{M}_{n\times n}(\mathbb{R}) : x_{ij} = x_{ji}$ for all relevant $(i, j)\}$. Show that V_2 is a subspace of $\mathcal{M}_{n\times n}(\mathbb{R})$—this is the space of real symmetric matrices—and find a basis for it.

(c) Let $V_3 := \{(x_{ij}) \in \mathcal{M}_{n\times n}(\mathbb{R}) : x_{ij} = -x_{ji}$ for all relevant $(i, j)\}$. Show that V_3 is a subspace of $\mathcal{M}_{n\times n}(\mathbb{R})$—this is the space of real *skew-symmetric* $n \times n$ matrices—and find a basis for it.