# Linear Algebra I, Sheet 4, MT2018
## Bases and subspaces. Dimension formula. Direct sums.

**1.** Find the row rank of each of the following matrices:

$$\text{(a)} \begin{pmatrix} 2 & 4 & -3 & 0 \\ 1 & -4 & 3 & 0 \\ 3 & -5 & 2 & 1 \end{pmatrix} \qquad \text{(b)} \begin{pmatrix} 1 & 2 & 3 & 0 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 5 & 2 \end{pmatrix} \qquad \text{(c)} \begin{pmatrix} 1 & 2 & 3 & 0 \\ 2 & 3 & 4 & 2 \\ 3 & 4 & 5 & 2 \end{pmatrix}$$

**2.** Let $V$ be a vector space and let $U$, $W$ be subspaces of $V$. Show that $U \setminus W$ is never a subspace. Find a necessary and sufficient condition for $U \cup W$ to be a subspace.

**3.** Let $V = \mathbb{R}^n$ where $n \geqslant 2$, and let $U$ and $W$ be subspaces of $V$ of dimension $n - 1$.

(a) Show that if $U \neq W$ then $\dim(U \cap W) = n - 2$.

(b) Now suppose that $n \geqslant 3$ and let $U_1$, $U_2$, $U_3$ be three distinct subspaces of dimension $n - 1$. Must it be true that $\dim(U_1 \cap U_2 \cap U_3) = n - 3$? Give a proof or find a counterexample.

(c) Show that if $\dim U \leqslant n - 2$ then there are infinitely many different subspaces $X$ such that $U \leqslant X \leqslant V$.

**4.** Let $V = \mathbb{R}^4$, and let

$$X = \{(x_1, x_2, x_3, x_4) \in V : x_2 + x_3 + x_4 = 0\},$$
$$Y = \{(x_1, x_2, x_3, x_4) \in V : x_1 + x_2 = x_3 - 2x_4 = 0\}.$$

Find bases for $X$, $Y$, $X \cap Y$ and $X + Y$, and write down their dimensions.

**5.** Let $V$ be an $n$-dimensional vector space.

(a) Prove that $V$ contains a subspace of dimension $r$ for each $r$ in the range $0 \leqslant r \leqslant n$.

(b) Show that if $U_0 \subsetneqq U_1 \subsetneqq \cdots \subsetneqq U_k \leqslant V$ (strict containments of subspaces of $V$) then $k \leqslant n$. Show also that if $k = n$ then $\dim U_r = r$ for $0 \leqslant r \leqslant k$.

(c) Let $U$, $W$ be subspaces of $V$ such that $U \leqslant W$. Show that there is a subspace $X$ of $V$ such that $W \cap X = U$ and $W + X = V$.

**6.** (a) Show that $V = U \oplus W$ if and only if every vector $v \in V$ can be expressed uniquely in the form $v = u + w$ with $u \in U$ and $w \in W$.

(b) Let $V = \mathbb{R}^3$ and $U = \{(x_1, x_2, x_3) \in V : x_1 + x_2 + x_3 = 0\}$. For each of the following subspaces $W$, either prove that $V = U \oplus W$, or explain why this is not true:

(i) $W = \{(x, 0, -x) : x \in \mathbb{R}\}$;

(ii) $W = \{(x, 0, x) : x \in \mathbb{R}\}$;

(iii) $W = \{(x_1, x_2, x_3) \in V : x_1 = x_3\}$.