Linear Algebra I, Sheet 5, MT2018

Linear transformations. Kernel and image. Rank and nullity.

1. Which of the following formulae describe linear transformations $T : \mathbb{R}^3 \to \mathbb{R}^3$?

(i) $T(x, y, z) = (y, z, 0)$;

(ii) $T(x, y, z) = (|x|, -z, 0)$;

(iii) $T(x, y, z) = (x - 1, x, y)$;

(iv) $T(x, y, z) = (yz, zx, xy)$.

As always, justify your answers.

2. Describe the kernel and image of each of the following linear transformations, and in each case find the nullity and the rank.

(i) $T : \mathbb{R}^4_{\text{col}} \to \mathbb{R}^3_{\text{col}}$ given by $T(X) = AX$ for $X \in \mathbb{R}^4_{\text{col}}$, where $A = \begin{pmatrix} 1 & -1 & 1 & 1 \\ 1 & 2 & -1 & 1 \\ 0 & 3 & -2 & 0 \end{pmatrix}$.

(ii) $V = \mathcal{M}_{n\times n}(\mathbb{R})$, and $T : V \to \mathbb{R}$ is given by $T(X) = \text{tr}(X)$, the sum $x_{11} + x_{22} + \cdots + x_{nn}$ of the entries on the main diagonal of X.

3. Let $V = \mathbb{R}_n[x]$, the vector space of real polynomials of degree $\leqslant n$. Define $D : V \to V$ to be differentiation with respect to x. Find the rank and nullity of D.

4. Let V be a finite-dimensional vector space, and let $S, T : V \to V$ be linear transformations.

(i) Show that $\text{Im}(S + T) \leqslant \text{Im}\, S + \text{Im}\, T$. Deduce that $\text{rank}(S + T) \leqslant \text{rank}\, S + \text{rank}\, T$.

(ii) Show that $\text{null}(ST) \leqslant \text{null}\, S + \text{null}\, T$. [Hint: Focus on the restriction of S to $\text{Im}\, T$, and consider its image and kernel.]

5. Let V be a finite-dimensional vector space.

(a) Let U, W be subspaces such that $V = U \oplus W$. Let $P : V \to V$ be the projection onto U along W, and let $Q : V \to V$ be the projection onto W along U.

(i) Show that $Q = i_V - P$.

(ii) Show that $P^2 = P$, that $Q^2 = Q$ and that $PQ = QP = 0$.

(b) Now let $T : V \to V$ be a linear transformation such that $T^2 = T$ (such linear transformations are said to be *idempotent*).

(i) For $v \in V$ let $u = Tv$ and let $w = v - Tv$. Show that $u \in \text{Im}\, T$, $w \in \ker T$ and $v = u + w$.

(ii) Show that $\text{Im}\, T \cap \ker T = \{0\}$.

(iii) Deduce that $V = U \oplus W$ where $U := \text{Im}\, T$, $W := \ker T$, and that T is the projection onto U along W.

6. Let V be an n-dimensional vector space and let $T : V \to V$ be a linear transformation. Prove that the following statements are equivalent:

(a) $\text{Im}\, T = \ker T$, and (b) $T^2 = 0$, n is even and $\text{rank}\, T = \frac{1}{2}n$.