Linear Algebra I, Sheet 6, MT2018

Matrices of linear transformations, change of basis, rank

1. Let V be a finite-dimensional vector space, and let U, W be subspaces such that $V = U \oplus W$. Let $T : V \to V$ be a linear transformation with the property that $T(U) \subseteq U$ and $T(W) \subseteq W$.

(a) Show that the matrix of T with respect to a basis of V which is the union of bases of U and of W has block form $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, where A is the matrix of the restriction of T to U and B is the matrix of the restriction of T to W.

(b) Now let $V = \mathbb{R}^4$ and define $T(x_1, x_2, x_3, x_4) = (x_1 + x_2, x_1 - x_2, x_4, 0)$. Find 2-dimensional subspaces U and W such that $T(U) \subseteq U$, $T(W) \subseteq W$, and $V = U \oplus W$.

(c) For the subspaces U, W you found in (b), find bases B_U for U and B_W for W, and find the matrix of T with respect to $B_U \cup B_W$ as in (a).

2. Let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$. Show that if $A^2 = 0$ then $\operatorname{rank} A \leqslant \frac{1}{2}n$. Show more generally that if $A^k = 0$ then $\operatorname{rank} A \leqslant n(k-1)/k$.

3. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map $T(x, y, z) = (y, -x, z)$ for all $(x, y, z) \in \mathbb{R}^3$. Let $\mathcal{E}$ be the standard basis of $\mathbb{R}^3$, and let $\mathcal{F}$ be the basis f_1, f_2, f_3 where $f_1 = (1, 1, 1)$, $f_2 = (1, 1, 0)$ and $f_3 = (1, 0, 0)$.

(a) Calculate the matrix A of T with respect to the standard basis $\mathcal{E}$ of $\mathbb{R}^3$.

(b) Calculate (directly) the matrix B of T with respect to the basis $\mathcal{F}$.

(c) Let I be the identity map of $\mathbb{R}^3$. Calculate the matrix P of I with respect to the bases $\mathcal{E}$, $\mathcal{F}$ and the matrix Q of I with respect to the bases $\mathcal{F}$, $\mathcal{E}$, and check that $PQ = I_3$.

(d) What do you predict is true about QBP? Now compute it—were you right?

4.

(a) Show that if X, $Y \in \mathcal{M}_{n\times n}(\mathbb{R})$ then $\operatorname{tr}(XY) = \operatorname{tr}(YX)$.

(b) Deduce that if A and B are similar $n \times n$ matrices then $\operatorname{tr}(A) = \operatorname{tr}(B)$.

5. Let $A \in \mathcal{M}_{m\times n}(\mathbb{R})$ and let $b \in \mathbb{R}^m_{\text{col}}$. Prove that

(a) if $m < n$ then the system of linear equations $Ax = 0$ always has a non-trivial solution;

(b) if $m < n$ then the system of linear equations $Ax = b$ has either no solution or infinitely many different solutions;

(c) if A has rank m then the system of linear equations $Ax = b$ always has a solution;

(d) if A has rank n then the system of linear equations $Ax = b$ has at most one solution;

(e) if $m = n$ and A has rank n, then the system $Ax = b$ has precisely one solution.

6. Let $n \geqslant 2$ and let $V_n := \{a_0 x^n + a_1 x^{n-1} y + \cdots + a_n y^n : a_0, a_1, \ldots, a_n \in \mathbb{R}\}$, the real vector space of homogeneous polynomials of degree n in two variables x and y. Let B_n be the "natural" ordered basis x^n, $x^{n-1}y$, $\ldots$, xy^{n-1}, y^n of V_n. Define $L : V_n \to V_{n-2}$ by $L(f) := \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$.

(a) Check that L (the two-variable Laplace operator) is a linear transformation.

(b) Find the matrix of L with respect to the bases B_n of V_n and B_{n-2} of V_{n-2}.

(c) Find the rank of this matrix, and hence find the dimension of $\{f \in V_n : L(f) = 0\}$.