Linear Algebra I, Sheet 7, MT2018

Bilinear forms and inner product spaces

1. Let V be a real vector space. A bilinear form $\langle -,- \rangle$ on V is said to be *skew-symmetric* if $\langle v_1, v_2 \rangle = -\langle v_2, v_1 \rangle$ for all $v_1,\ v_2 \in V$. It is said to be *alternating* if $\langle v, v \rangle = 0$ for all $v \in V$.

(a) Show that every bilinear form on V may be written uniquely as the sum of a symmetric bilinear form and a skew-symmetric bilinear form.

(b) Show that a bilinear form $\langle -,- \rangle$ on V is alternating if and only if it is skew-symmetric.

2. Let V be a real vector space and let $\langle -,- \rangle$, $\langle -,- \rangle_1$ and $\langle -,- \rangle_2$ be inner products on V. The *norm* of a vector v is defined by $\|v\| := \sqrt{\langle v, v \rangle}$. Norms $\|v\|_1$ and $\|v\|_2$ are defined analogously.

(a) Show that if $u,\ v \in V$ then $\langle u, v \rangle = \frac{1}{2}(\|u+v\|^2 - \|u\|^2 - \|v\|^2)$.

(b) Deduce that if $\|x\|_1 = \|x\|_2$ for all $x \in V$ then $\langle u, v \rangle_1 = \langle u, v \rangle_2$ for all $u,\ v \in V$.

3. (a) Let V be a 2-dimensional real vector space with basis $\{e_1, e_2\}$. Describe all the inner products $\langle -,- \rangle$ on V for which $\langle e_1, e_1 \rangle = 1$ and $\langle e_2, e_2 \rangle = 1$.

(b) Show that if V is the 3-dimensional real vector space $\mathbb{R}_2[x]$ of polynomials of degree at most 2 in x, then the definition $\langle f(x), g(x) \rangle := f(0)g(0) + f(1)g(1) + f(2)g(2)$ describes an inner product on V.

4. Let V be an n-dimensional real vector space with inner product $\langle -,- \rangle$, and let U be an m-dimensional subspace of V. Define $U^\perp := \{v \in V : \langle v, u \rangle = 0 \text{ for all } u \in U\}$.

(a) Show that $U^\perp$ is a subspace of V, and that $U^\perp \cap U = \{0\}$.

(b) Let $u_1,\ \ldots,\ u_m$ be a basis for U. Define $T : V \to \mathbb{R}^m$ by $T(v) = (x_1, \ldots, x_m)$, where $x_i := \langle v, u_i \rangle$ for $i = 1,\ \ldots,\ m$. Show that T is a linear transformation.

(c) For T as in (b), show that $\ker T = U^\perp$ and that $\operatorname{rank} T = m$.

(d) Deduce that $\dim U^\perp = n - m$ and that $V = U \oplus U^\perp$.

(e) Let $V := \mathbb{R}_2[x]$ with the inner product defined in Q3(b), and let U be the subspace spanned by 1 and x. Find polynomials $p(x) \in U$ and $q(x) \in U^\perp$ such that $x^2 = p(x) + q(x)$.

5. Let $a,\ b,\ c$ be real numbers. Use the Cauchy-Schwarz inequality to show that if $x^2 + y^2 + z^2 = 1$ then $ax + by + cz \leqslant \sqrt{a^2 + b^2 + c^2}$. At what points does equality hold?

6. Let V be a complex vector space, and let $\langle -,- \rangle : V \times V \to \mathbb{C}$ be a *sesquilinear* form: that is, for all $u,\ v,\ w \in V$ and all $\alpha,\ \beta \in \mathbb{C}$,

$$\text{(i)} \quad \langle \alpha u + \beta v, w \rangle = \alpha \langle u, w \rangle + \beta \langle v, w \rangle \qquad \text{and} \qquad \text{(ii)} \quad \langle u, v \rangle = \overline{\langle v, u \rangle}.$$

Note that $\langle v, v \rangle \in \mathbb{R}$ for all $v \in V$. If $\langle v, v \rangle > 0$ for all $v \in V \setminus \{0\}$ then the form is said to be *Hermitian*.

(a) Show that $\langle u, \alpha v + \beta w \rangle = \overline{\alpha} \langle u, v \rangle + \overline{\beta} \langle u, w \rangle$ for all $u,\ v,\ w \in V$ and all $\alpha,\ \beta \in \mathbb{C}$.

(b) Show that if $\langle (x_1, \ldots, x_n), (y_1, \ldots, y_n) \rangle = x_1 \overline{y_1} + \cdots + x_n \overline{y_n}$ then $\langle -,- \rangle$ is a Hermitian form on $\mathbb{C}^n$.

(c) Now suppose that V is finite-dimensional over $\mathbb{C}$ and that the form $\langle -,- \rangle$ is Hermitian. Let U be a subspace of V, and, as in Q4, define $U^\perp := \{v \in V : \langle v, u \rangle = 0 \text{ for all } u \in U\}$. Show that $U^\perp \leqslant V$ and that $V = U \oplus U^\perp$.