Complex Numbers Problems Sheet, MT2018

Solutions for this sheet will be available on the course materials webpage from Monday of Week 2. Students are encouraged to discuss the problems and their solutions with each other.

1. Which of the following quadratic equations have real roots, which do not?

$$3x^2 + 2x - 1 = 0; \quad 2x^2 - 6x + 9 = 0; \quad -4x^2 + 7x - 9 = 0; \quad 4x^2 - 9x + 5 = 0.$$

2. Write a careful proof of the theorem that if $z,\, w \in \mathbb{C}$ then $\overline{z+w} = \overline{z} + \overline{w}$ and $\overline{zw} = \overline{z}\,\overline{w}$.

3. Put each of the following complex numbers into standard form $a + bi$:

$$(1+2i)(3-i); \quad (2+i)(1-2i); \quad (1+i)^4; \quad (1-\sqrt{3}i)^3; \quad \frac{7-2i}{5+12i}; \quad \frac{i}{1-i}.$$

4. Find the modulus and argument of each of the following complex numbers:

$$1+\sqrt{3}i; \quad (2+i)(3-i); \quad (1+i)^5.$$

5. On separate Argand diagrams sketch each of the following subsets of $\mathbb{C}$:

$A := \{z : |z| < 1\}$; $\quad B := \{z | \operatorname{Re} z = 3\}$; $\quad C := \{z : -\frac{\pi}{4} < \arg z < \frac{\pi}{4}\}$;
$D := \{z : \arg(z - i) = \frac{\pi}{2}\}$; $\quad E := \{z : |z - 3 - 4i| = 5\}$; $\quad F := \{z : |z - 1| = |z - i|\}$.

6. For each of the following complex numbers w, what transformation of the Argand diagram does multiplication by w represent?

$$i; \quad (1+i); \quad (1-i); \quad (3+4i).$$

7. Use De Moivre's Theorem to show that if $\theta \in \mathbb{R}$ then

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta; \quad \sin 5\theta = (16\cos^4\theta - 12\cos^2\theta + 1)\sin\theta.$$

8. Write down the primitive 6^{th} roots of unity and the primitive 8^{th} roots of unity in standard form $a + bi$.

9. Let $\phi := \cos(2\pi/5) + i\sin(2\pi/5)$, a primitive 5^{th} root of 1. Define $\alpha := \phi + \phi^4$, $\beta := \phi^2 + \phi^3$.

(i) Show that α and β are real numbers.

(ii) Show that $\alpha + \beta = -1$ and $\alpha\beta = -1$ (so that α, β are the roots of $x^2 + x - 1 = 0$).

(iii) Deduce that $\cos(2\pi/5) = \frac{1}{4}(\sqrt{5} - 1)$.

10. Find the square roots of $-7 + 24i$. Now solve the equation $z^2 - (2+2i)z + (7-22i) = 0$.

11. By considering the seventh roots of -1, show that

$$\cos\frac{\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{5\pi}{7} = \frac{1}{2}.$$

What is the value of

$$\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7}?$$

12. Show that $\sigma^8 = -1$ if and only if σ is a primitive 16^{th} root of 1. Find the roots of the equation $x^8 = -1$, and use them to write $x^8 + 1$ as the product of four quadratic factors with real coefficients.