Prelim Introduction to University Mathematics, Michaelmas Term 2018:
Exercise Sheet 1: *Sets, relations, functions.*

0. Ensure that you know the Greek alphabet. [Lecture Notes, appendix]

1. Let A, B, C be sets. Write out a proof that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

2. Write out proofs of De Morgan's Laws: if X is a set and A, B are subsets of it then

$$X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B) \quad \text{and} \quad X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B).$$

3. Let A and B be finite sets.

(i) Assume first A and B are disjoint (for part (i) only). Use induction on $|B|$ to show that $A \cup B$ is finite and $|A \cup B| = |A| + |B|$.

(ii) Show that $A \cup B = (A \setminus B) \cup (A \cap B) \cup (B \setminus A)$. Deduce that

$$|A| + |B| = |A \cup B| + |A \cap B|.$$

4. Let A be a set and B_i a family of sets indexed by a non-empty set I. Show that

$$A \cap \Big(\bigcup_{i \in I} B_i\Big) = \bigcup_{i \in I} (A \cap B_i) \quad \text{and} \quad A \cup \Big(\bigcap_{i \in I} B_i\Big) = \bigcap_{i \in I} (A \cup B_i).$$

5. Which of the following relations on $\mathbb{N}$ are reflexive, which are symmetric, which are transitive?

(i) the relation $a \mid b$ (read as 'a divides b');
(ii) the relation $a \nmid b$ (does not divide);
(iii) a, b are related if a, b leave the same remainder after division by 2016;
(iv) a, b are related if $\operatorname{hcf}(a, b) > 2016$.
[Here $\operatorname{hcf}(a, b)$ denotes the highest common factor, also known as the greatest common divisor, of a and b. It is the largest natural number that divides both a and b.]

6. How many partitions are there of a set of size 1? of size 2? of size 3? of size 4? of size 5?

7. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) := \sin x$ for all real numbers x.

(i) What are $f([0, \pi])$, $f([0, 2\pi])$, $f([0, 3\pi])$?
(ii) What are $f^{-1}(\{0\})$, $f^{-1}(\{1\})$, $f^{-1}(\{2\})$?
(iii) Let $A := [0, \pi]$ and $B := [2\pi, 3\pi]$. Show that $f(A \cap B) \neq f(A) \cap f(B)$.
(iv) Let $A := [0, \pi]$. Find $f(A)$, $f^{-1}(f(A))$, and $f(f^{-1}(A))$.

8. Let X, Y be sets and $f : X \to Y$, let $A, B \subseteq X$ and let C, $D \subseteq Y$. Prove that:

(i) $f(A) \cup f(B) = f(A \cup B)$;
(ii) $f^{-1}(C) \cup f^{-1}(D) = f^{-1}(C \cup D)$;
(iii) $f^{-1}(C) \cap f^{-1}(D) = f^{-1}(C \cap D)$.

p.t.o.

9. Let $f : A \to B$, where A, B are sets.

(i) Show that if $X \subseteq A$ then $X \subseteq f^{-1}(f(X))$. Must equality hold?

(ii) Is it always true that if $Y \subseteq B$ then $Y \subseteq f(f^{-1}(Y))$?

10. Let $A := \{0, 1, 2\}$ and $B := \{0, 1, 2, 3, 4\}$.

(i) How many $f : A \to B$ are there?

(ii) How many $f : B \to A$ are there?

(iii) How many injective $f : A \to B$ are there?

(iv) How many injective $f : B \to A$ are there?

(v) How many surjective $f : A \to B$ are there?

(vi) How many surjective $f : B \to A$ are there?

[HINT: for part (vi) you might find your analysis of partitions of a set of size 5 useful—see Qn 6.]