Prelim Introduction to University Mathematics, Michaelmas Term 2018:

Exercise Sheet 2: *The language of mathematics; proofs and refutations; problem-solving.*

1. Explain what is wrong with the following (distressingly common) attempted proof of the AM-GM Inequality for two non-negative real numbers.

> We are given that $a \geqslant 0$, $b \geqslant 0$. Suppose that $\sqrt{ab} \leqslant \frac{1}{2}(a+b)$. Then $2\sqrt{ab} \leqslant a+b$, so $4ab \leqslant (a+b)^2$. Expanding and tidying we see that $0 \leqslant a^2 - 2ab + b^2$, that is, $0 \leqslant (a-b)^2$. Since squares are always positive this is true. Therefore $\sqrt{ab} \leqslant \frac{1}{2}(a+b)$, which is the AM-GM Inequality.

Show how to turn it into a correct argument.

2. Which of the following statements about natural numbers are true, which false?

(a) 2 is prime or 2 is odd.

(b) 2 is prime or 2 is even.

(c) If 2 is odd then 2 is prime.

(d) If 2 is even then 2 is prime.

(e) For all $n \in \mathbb{N}$, if n is a square number then n is not prime.

(f) For all $n \in \mathbb{N}$, n is not prime if and only if n is a square number.

(g) For all even primes $p > 2$, $p^2 = 2016$.

3. Each card in a pack has a number on one side and a letter on the other. Four cards are placed on the table:

2	3	A	B

You are permitted to turn just two cards over in order to test the following hypothesis: *a card that has an even number on one side has a vowel on the other*. Which two cards should you turn? Or is it impossible?

4. Formulate Mathematical Induction using the symbols $\forall$ and $\Rightarrow$.

5. Let $f : \mathbb{R} \to \mathbb{R}$. Translate the following formula into English:

$$\forall a \in \mathbb{R} : \forall \varepsilon \in \mathbb{R}^{>0} : \exists \delta \in \mathbb{R}^{>0} : \forall x \in \mathbb{R} : |x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon \, .$$

6. Let $f : \mathbb{R} \to \mathbb{R}$. Express the following using the symbols $\forall$ and $\Rightarrow$:

> for all real numbers a and x and for every positive real number ε there exists a positive real number δ such that $|f(x) - f(a)| < \varepsilon$ whenever $|x - a| < \delta$.

Does it say much the same as the formula in Qn 5?

7. Organise the following assertions about a function $f : \mathbb{R} \to \mathbb{R}$ into pairs such that one member of the pair is true if and only if the other is false:

(1) for all real numbers x, y, there exists a real number $z > x$ such that $f(z) > y$;

(2) for every real number x there exists a real number y such that for all real z either $z \leqslant y$ or $f(z) \neq x$;

(3) there exist real numbers x, y such that for every real number z if $z \leqslant y$ then $f(z) = x$;

(4) there is a real number x such that for every real number y there is a real number $z > y$ for which $f(z) = x$;

(5) for all real x and y there is a real number z such that $z \leqslant y$ and $f(z) \neq x$.

(6) There exist real numbers x and y such that for every real number z either $z \leqslant x$ or $f(z) \leqslant y$.

8. Let's write $n =$ '$d_m d_{m-1} \cdots d_2 d_1 d_0$' where $0 \leqslant d_i \leqslant 9$ for $0 \leqslant i \leqslant m$ and $d_m \neq 0$, to mean that n is an $(m+1)$-digit natural number and the given string of digits is its decimal representation. Prove that n and $d_0 + d_1 + \cdots + d_{m-1} + d_m$ leave the same remainder when divided by 9.

9. Does there exist a positive integer N which is a power of 2, and a different positive integer M obtained from N by permuting its digits (in the usual base 10 representation), such that M is also a power of 2? Note that we do not allow the base 10 representation of a positive integer to begin with 0.