Fourier series & PDEs

Motivation

Example: existence of a convergent Fourier series

- Recall $e^{z} = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ for $z \in \mathbb{C}$.
- If we let $z = e^{i\theta} = \cos\theta + i\sin\theta$, where $\theta \in \mathbb{R}$, then

$$\mathrm{Im}(e^{z}) = \mathrm{Im}(e^{\cos\theta} e^{i\sin\theta}) = e^{\cos\theta}\sin(\sin\theta),$$

$$\mathrm{Im}(z^n) = \mathrm{Im}(e^{in\theta}) = \sin(n\theta).$$

- Hence, $e^{\cos\theta}\sin(\sin\theta) = \underbrace{\sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n!}}_{\text{Fourier (sine) series}}$ for $\theta \in \mathbb{R}$.

Example: heat conduction

- Suppose $T(x,t)$ s.t.
  ① $T_t = T_{xx}$ for $0 < x < \pi$, $t > 0$
  ② $T(0,t) = 0$, $T(\pi,t) = 0$ for $t > 0$
  ③ $T(x,0) = e^{\cos x}\sin(\sin x)$ for $0 < x < \pi$.
- Observe $T(x,t) = \sum_{n=1}^{N} b_n \sin(nx) e^{-n^2 t}$ satisfies ① and ② for all $b_1, b_2, \dots, b_N \in \mathbb{R}$, $N \in \mathbb{N}\setminus\{0\}$.
- $Q^{\underline{n}}$: how should we pick N and the constants b_n?

  $A^{\underline{ns}}$: $N = \infty$ and $b_n = \frac{1}{n!}$ to satisfy ③, i.e. a sol$^{\underline{n}}$ of the IBVP ① – ③ is $T(x,t) = \sum_{n=1}^{\infty} \frac{1}{n!}\sin(nx)e^{-n^2 t}$.

- But what about other initial conditions?

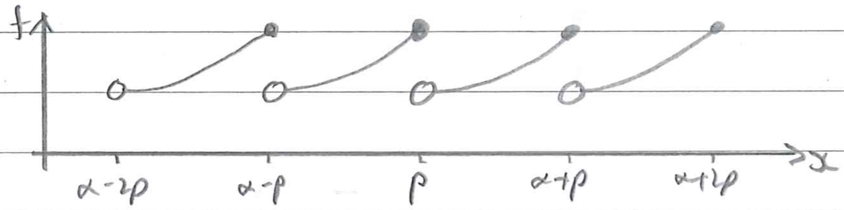
②

Periodic, even and odd functions

- <u>Defⁿ</u>: $f : \mathbb{R} \to \mathbb{R}$ is a periodic function if $\exists p > 0$ s.t. $f(x+p) = f(x)$ $\forall x \in \mathbb{R}$. In this case p is a period for f and f is called p-periodic. Period is not unique, but if $\exists$ smallest such p, it is called the prime period.

- E.g. $f = $ const. is p-periodic $\forall p > 0$, so has no prime period.
  $\sin x$ has prime period 2π.
  x & x^2 not periodic.

- Note graph of f repeats every p, e.g.



- $f : (\alpha, \alpha+p] \to \mathbb{R}$ can be extended uniquely to be p-periodic.

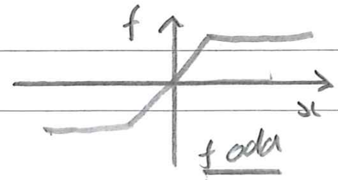
- <u>Defⁿ</u>: The periodic extension $F : \mathbb{R} \to \mathbb{R}$ of $f : (\alpha, \alpha+p] \to \mathbb{R}$ is defined by $F(x) = f(x - mp)$, where for each $x \in \mathbb{R}$, m is the unique integer s.t. $x - mp \in (\alpha, \alpha+p]$.

- f, g p-periodic $\Rightarrow$ (1) f, g np-periodic $\forall n \in \mathbb{N} \setminus \{0\}$
  (2) $\alpha f + \beta g$ p-periodic $\forall \alpha, \beta \in \mathbb{R}$
  (3) fg p-periodic
  (4) $f(\lambda x)$ p/λ-periodic $\forall \lambda > 0$
  (5) $\int_0^p f(x)\,dx = \int_\alpha^{\alpha+p} f(x)\,dx$ $\forall \alpha \in \mathbb{R}$

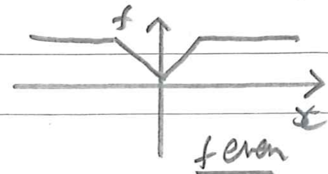
- <u>Defⁿ</u>: $f : \mathbb{R} \to \mathbb{R}$ odd if $f(x) = -f(-x)$ $\forall x \in \mathbb{R}$
  $f : \mathbb{R} \to \mathbb{R}$ even if $f(x) = f(x)$ $\forall x \in \mathbb{R}$

- E.g. x^n odd for n odd, even for n even (hence names)

- Note symmetries of graphs:



f odd



f even

- Properties of odd/even functions: If f, f_1 odd and g, g_1 even, then (1) $f(0) = 0$;
  (2) $\int_{-\alpha}^{\alpha} f(x)\,dx = 0 \quad \forall \alpha \in \mathbb{R}$;
  (3) $\int_{-\alpha}^{\alpha} g(x)\,dx = 2\int_{0}^{\alpha} g(x)\,dx \quad \forall x \in \mathbb{R}$;
  (4) fg odd, ff_1 even, gg_1 even.

Fourier series for functions of period 2π

- Let $f : \mathbb{R} \to \mathbb{R}$ be a periodic function of period 2π.
- We want an expansion for f of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)) \qquad (*)$$

- Q1: If $(*)$ is true, can we find the constants a_n, b_n in terms of f?

  Q2: With these a_n & b_n, when is $(*)$ true?

Question 1

- Suppose $(*)$ true and we can integrate it term by term, then

$$\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2}a_0 \int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty}\Big(a_n \underbrace{\int_{-\pi}^{\pi} \cos(nx)\,dx}_{=0} + b_n \underbrace{\int_{-\pi}^{\pi} \sin(nx)\,dx}_{=0}\Big)$$

$\Rightarrow$ $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$, i.e. $\frac{a_0}{2}$ is the mean of f over a period.

- Lemma: Let $m, n \in \mathbb{N}\setminus\{0\}$. Then we have the orthogonality relations:

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx = \pi\delta_{mn}, \quad \int_{-\pi}^{\pi} \cos(mx)\sin(nx)\,dx = 0, \quad \int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx = \pi\delta_{mn},$$

where δ_{mn} is Kronecker's delta, i.e. $\delta_{mn} = \begin{cases} 1 & \text{for } m = n, \\ 0 & \text{for } m \neq n. \end{cases}$

Pf: see online notes and sheet 1.

④

- Fix $m \in \mathbb{N}\setminus\{0\}$, multiply (*) by $\cos(mx)$ and assume $\int\sum = \sum\int$

$$\Rightarrow \int_{-\pi}^{\pi} f(x)\cos(mx)\,dx = \frac{1}{2}a_0\int_{-\pi}^{\pi}\cos(mx)\,dx$$

$$+\sum_{n=1}^{\infty}\left(a_n\int_{-\pi}^{\pi}\cos(mx)\cos(nx)\,dx + b_n\int_{-\pi}^{\pi}\cos(mx)\sin(nx)\,dx\right)$$

$$= \frac{1}{2}a_0\cdot 0 + \sum_{n=1}^{\infty}\left(a_n\pi\delta_{mn} + b_n\cdot 0\right)$$

$$= \pi a_m$$

$$\Rightarrow \quad \underline{\underline{a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx \quad \text{for } m \in \mathbb{N}\setminus\{0\}}}$$

- Similarly, we can fix $m \in \mathbb{N}\setminus\{0\}$, multiply (*) by $\sin(mx)$ and assume $\int\sum = \sum\int$ to get

$$\underline{\underline{b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)\,dx \quad \text{for } m \in \mathbb{N}\setminus\{0\}}}$$

- <u>Defn</u>: Suppose f is such that the <u>Fourier coefficients</u>

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \quad (n \in \mathbb{N}), \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx \quad (n \in \mathbb{N}\setminus\{0\})$$

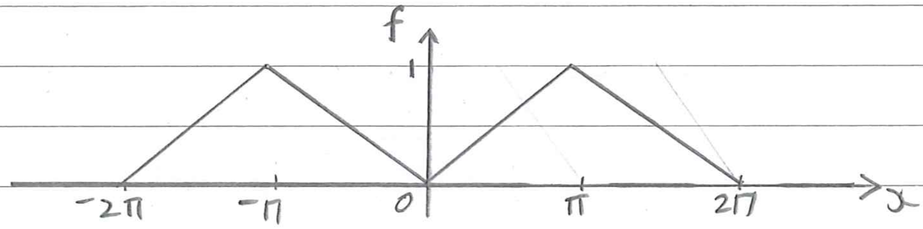
exist. Then we write

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n\cos(nx) + b_n\sin(nx)\right),$$

where $\sim$ means the RHS is the <u>Fourier series for f</u>, regardless of whether or not it converges to f.

- Note factor of $\frac{1}{2}$ in first term is for algebraic convenience.

- **Example 2.1**: Find the Fourier series (FS) for the 2π-periodic function f defined by $f(x) = |x|$ for $-\pi < x \leq \pi$.



$f(x)$ even $\Rightarrow$ $f(x)\cos(nx)$ even and $f(x)\sin(nx)$ odd

$$\Rightarrow \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx, \; b_n = 0$$

Calculate $a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = \left[\frac{2}{\pi}\frac{x^2}{2}\right]_0^{\pi} = \pi.$

For $n > 0$, we use integration by parts:

$$(uv)' = u'v + uv' \Rightarrow \quad [uv]_a^b = \int_a^b u'v + uv'\,dx$$

Pick $u = x$, $v = \frac{1}{n}\sin(nx)$, $a = 0$, $b = \pi$

$$\Rightarrow \left[\frac{x}{n}\sin(nx)\right]_0^{\pi} = \int_0^{\pi} 1\cdot\frac{1}{n}\sin(nx) + x\cos(nx)\,dx$$

$$\Rightarrow \int_0^{\pi} x\cos(nx)\,dx = -\int_0^{\pi}\frac{1}{n}\sin(nx)\,dx = \left[\frac{\cos(nx)}{n^2}\right]_0^{\pi} = \frac{(-1)^n - 1}{n^2}$$

$$\Rightarrow a_n = -\frac{2(1-(-1)^n)}{\pi n^2} = \begin{cases} 0 & \text{for } n = 2m,\ m \in \mathbb{N}\setminus\{0\} \\ -\frac{4}{\pi(2m+1)^2} & \text{for } n = 2m+1,\ m \in \mathbb{N} \end{cases}$$

Hence,

$$f(x) \sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{\cos((2m+1)x)}{(2m+1)^2} \qquad \square$$

- **Remarks**

(1) Partial sums defined by

$$S_N(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{m=0}^{N}\frac{\cos((2m+1)x)}{(2m+1)^2} \quad \text{for } N \in \mathbb{N}.$$

Plots in handout suggest that FS converges on $\mathbb{R}$, i.e.

$$\lim_{N\to\infty} S_N(x) = f(x) \quad \text{for } x \in \mathbb{R}. \qquad (+)$$

(2) If this is true, can pick x to evaluate the sum of a series, e.g. $x = 0$

$$\Rightarrow \quad 0 = \frac{\pi}{2} - \frac{4}{\pi}\sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} \Rightarrow \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} = \frac{\pi^2}{8}.$$

Sine and cosine series

- Let f be 2π-periodic and s.t. Fourier coefficients exist.
- $f(x)$ odd $\Rightarrow$ $f(x)\cos(nx)$ odd and $f(x)\sin(nx)$ even

$$\Rightarrow \quad a_n = 0, \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$$

$$\Rightarrow \quad f(x) \sim \sum_{n=1}^{\infty} b_n \sin(nx),$$ called a <u>Fourier sine series</u>

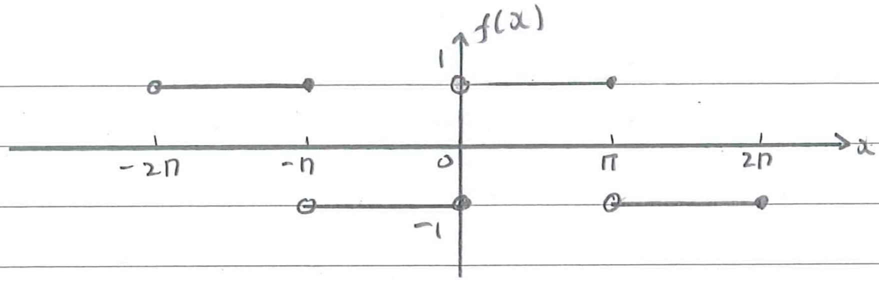
- Note this is also true if f is odd only for $x \neq k\pi$, $k \in \mathbb{Z}$.
- Similarly, f even $\Rightarrow$ $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$, called a <u>Fourier cosine series</u>, where $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx$.

Question 2

- When does the FS for f converge?

Example 2.2: Find the FS for the 2π-periodic function f defined by

$$f(x) = \begin{cases} 1 & \text{for } 0 < x \leq \pi, \\ -1 & \text{for } -\pi < x \leq 0. \end{cases}$$



$$f \text{ odd for } x \neq k\pi,\ k \in \mathbb{Z} \Rightarrow a_n = 0,\ b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$$

$$f(x) = 1 \text{ for } 0 < x < \pi \Rightarrow b_n = \left[\frac{-2}{\pi}\frac{\cos(nx)}{n}\right]_0^{\pi} = \frac{2(1-(-1)^n)}{\pi n}$$

$$\Rightarrow f(x) \sim \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{\sin((2m+1)x)}{2m+1} \qquad \square$$

Remarks

(1) Partial sums defined by

$$S_N(x) = \frac{4}{\pi}\sum_{m=0}^{N}\frac{\sin((2m+1)x)}{2m+1} \qquad \text{for } N \in \mathbb{N}.$$

Plots in handout suggest that

$$\lim_{N\to\infty} S_N(x) = \begin{cases} f(x) & \text{for } x \neq k\pi,\ k\in\mathbb{Z}, \\ 0 & \text{for } x = k\pi,\ k \in \mathbb{Z}. \end{cases} \qquad (\#)$$

(2) Note slower convergence than in example 2.1 and persistent overshoot near discontinuities of f — called Gibb's phenomenon (more later).

Convergence of Fourier series

- **Def$^{\underline{n}}$**: $f(c_+) = \lim\limits_{\substack{h\to 0\\ h>0}} f(c+h)$ if it exists (RH limit at c)

  $f(c_-) = \lim\limits_{\substack{h\to 0\\ h<0}} f(c+h)$ if it exists (LH limit at c)

- Remarks:

(1) $f(c)$ need not be defined for $f(c+)$ or $f(c-)$ to exist.

(2) Existence part important, e.g. $f(x) = \sin(\frac{1}{x})$ for $x \neq 0 \Rightarrow$ $f(0\pm)$ do not exist.

(3) $f(c+) = f(c-) = f(c) \Leftrightarrow f$ continuous at c.

(4) In example 2.2, f is continuous for $x \neq k\pi$, $k \in \mathbb{Z}$, with e.g. $f(0+) = 1$, $f(0-) = -1$, $f(\pi+) = -1$, $f(\pi-) = 1$.

- <u>Defn</u>: f is <u>piecewise continuous</u> on $(a,b) \subseteq \mathbb{R}$ if there exist a finite number of points $x_1, x_2, \ldots, x_m$ with $a = x_1 < x_2 < \ldots < x_m = b$ such that
  (i) f is defined and continuous on (x_k, x_{k+1}) $\forall k = 1, \ldots, m-1$;
  (ii) $f(x_k+)$ exists for $k = 1, \ldots, m-1$;
  (iii) $f(x_k-)$ exists for $k = 2, \ldots, m$.

- Note that f need not be defined at its exceptional points $x_1, \ldots, x_m$!

- E.g. the functions in examples 2.1 and 2.2 are piecewise continuous on any interval $(a,b) \subseteq \mathbb{R}$.

- <u>Fourier Convergence Theorem (FCT)</u>

Let f be 2π-periodic, with f and f' piecewise continuous on $(-\pi, \pi)$. Then, the Fourier coefficients a_n and b_n exist, and

$$\frac{1}{2}\left(f(x+) + f(x-)\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right)$$

for $x \in \mathbb{R}$.

- Discuss further next time, but note here that FCT $\Rightarrow$ (+) and (++) are true.