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## Remarks

(1) $f, f'$ piecewise continuous (p.c.) on $(-\pi, \pi) \Rightarrow \exists x_1, x_2, \ldots, x_m \in \mathbb{R}$ with $-\pi = x_1 < x_2 < \ldots < x_m = \pi$ such that

(i) $f$ and $f'$ are continuous on $(x_k, x_{k+1})$ for $k = 1, \ldots, m-1$;

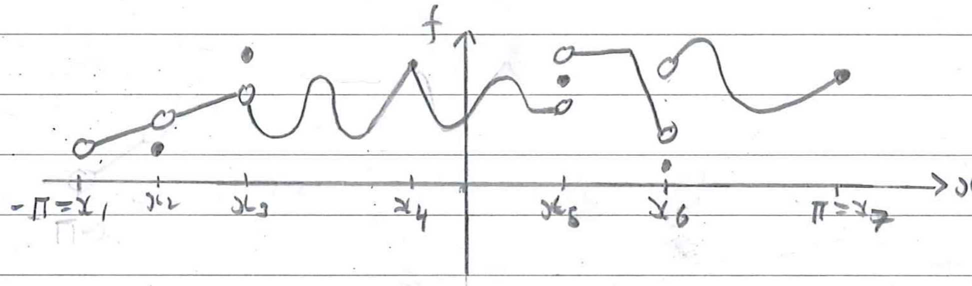
(ii) $f(x_{k+})$ and $f'(x_{k+})$ exist for $k = 1, \ldots, m-1$;

(iii) $f(x_{k-})$ and $f'(x_{k-})$ exist for $k = 2, \ldots, m$.

Orally

Thus, in any period $f, f'$ are continuous except possibly at a finite number of points; at each such point $f'$ need not be defined, and one or both of $f$ and $f'$ may have a jump discontinuity.

E.g.




E.g. $f(x) = \begin{cases} x^{1/2} & \text{for } 0 < x \le \pi \\ 0 & \text{for } -\pi < x \le 0 \end{cases} \Rightarrow f'(x) = \begin{cases} \frac{1}{2} x^{-1/2} & \text{for } 0 < x < \pi \\ 0 & \text{for } -\pi < x < 0 \\ \text{undefined} & \text{for } x = 0, \pi \end{cases}$

$\Rightarrow f$ p.c. on $(-\pi, \pi)$, but $f'$ not.

(2) Proof not examinable, but one method is as follows: firstly, show that

$$\frac{a_0}{2} + \sum_{n=1}^{N} \left(a_n \cos(nx) + b_n \sin(nx)\right) - \frac{1}{2}\left(f(x_+) + f(x_-)\right) = \int_0^{\pi} F(x,t) \sin\left((N+\tfrac{1}{2})t\right) dt,$$

where $F(x,t) = \frac{1}{\pi}\left(\frac{f(x+t) - f(x_+)}{t} + \frac{f(x-t) - f(x_-)}{t}\right)\left(\frac{t}{2\sin(t/2)}\right)$;

secondly, show $F(x,t)$ is a p.c. f$^n$ of $t$ on $(0, \pi)$, so that Riemann-Lebesgue Lemma (Analysis III) implies

$$\int_0^{\pi} F(x,t) \sin\left((N+\tfrac{1}{2})t\right) dt \to 0 \text{ as } N \to \infty.$$

(3) $f$ continuous at $x \Rightarrow \frac{1}{2}(f(x_+) + f(x_-)) = f(x)$.

(4) If $f$ defined only on e.g. $(-\pi, \pi]$, FCT holds for its $2\pi$-periodic extension.

(5) Can integrate termwise under weaker conditions, e.g. if $f$ is only $2\pi$-periodic and p.c. on $(-\pi, \pi)$, then FCT

$$\Rightarrow \int_0^X f(x)\,dx = \frac{1}{2}a_0 X + \sum_{n=1}^{\infty}\left(a_n \int_0^X \cos(nx)\,dx + b_n \int_0^X \sin(nx)\,dx\right)$$

for $X \in \mathbb{R}$; note that LHS is $2\pi$-periodic iff $a_0 = 0$.

(6) But need stronger conditions to differentiate termwise, e.g. if $f$ is $2\pi$-periodic and continuous on $\mathbb{R}$ with $f'$ and $f''$ p.c. on $(-\pi, \pi)$, then FCT

$$\Rightarrow \frac{1}{2}(f'(x_+) + f'(x_-)) = \sum_{n=1}^{\infty}\left(a_n \frac{d}{dx}(\cos(nx)) + b_n \frac{d}{dx}(\sin(nx))\right)$$

for $x \in \mathbb{R}$.

## Rate of convergence

- The smoother $f$, i.e. the more continuous derivatives, the faster the convergence of the FS for $f$.

- If the first jump discontinuity is in the $p$th derivative of $f$, with the convension that $p = 0$ if there is a jump discontinuity in $f$, then typically the non-zero $a_n$ and $b_n$ decay like $\frac{1}{n^{p+1}}$ as $n \to \infty$. E.g $p = 1$ in ex. 2.1, while $p = 0$ in ex. 2.2.

- Extremely useful result in practice (e.g. how many terms to keep for an accurate approximation) and for checking calculations. e.g. for $\approx 1\%$ accuracy need 100 terms for $p=1$, 10 for $p=0$.

(11)

## Gibb's phenomenon

- This is the persistent overshoot in ex. 2.2 near a jump discontinuity.
- Happens whenever $\exists$ a jump discontinuity.
- As # terms in partial sum $\to \infty$,

  width overshoot region $\to 0$ (by FCT)

  height overshoot region $\to \gamma |f(x_+) - f(x_-)|$,

  where $\gamma = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\sin x}{x}\,dx \approx 1.18$, i.e. $\approx 9\%$ (top & bottom.
- Awful for approximation purposes!

## Functions of any period

- Suppose now $f(x)$ is a periodic function of period $2L > 0$.
- Make transformation $x = \frac{LX}{\pi}$, $f(x) = g(X)$, then for $X \in \mathbb{R}$,

$$g(X+2\pi) \underset{\uparrow}{=} f\left(\tfrac{L}{\pi}(X+2\pi)\right) = f\left(\tfrac{LX}{\pi} + 2L\right) \underset{\uparrow}{=} f\left(\tfrac{LX}{\pi}\right) \underset{\uparrow}{=} g(X).$$

$(g(X) = f(\frac{L}{\pi}X))$ $\quad$ ($f$ $2L$-periodic) $\quad$ $(\frac{LX}{\pi} = x)$

Thus, $g$ is $2\pi$-periodic and we can use transformation to derive theory for $f$ from that for $g$ above.

- Here we summarize the key results.
- FS: $g(X) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nX) + b_n \sin(nX))$

$$\Leftrightarrow f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

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- <u>Fourier coefficients:</u>

$$a_n \underset{(\text{defn})}{=} \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos(nx)\,dx \underset{(x=\frac{\pi x}{L})}{=} \frac{1}{\pi}\int_{-L}^{L} g\left(\frac{\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)\frac{\pi\,dx}{L} \underset{(g(x)=f(x))}{=} \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

EOL3 Similarly, $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$

- <u>Important remark</u>: These formulae may be derived directly from the FS for $f$ by assuming $\int\sum = \sum\int$ and using the orthogonality relation

$$\int_{-L}^{L}\cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx = L\delta_{mn},$$

$$\int_{-L}^{L}\cos\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = 0,$$

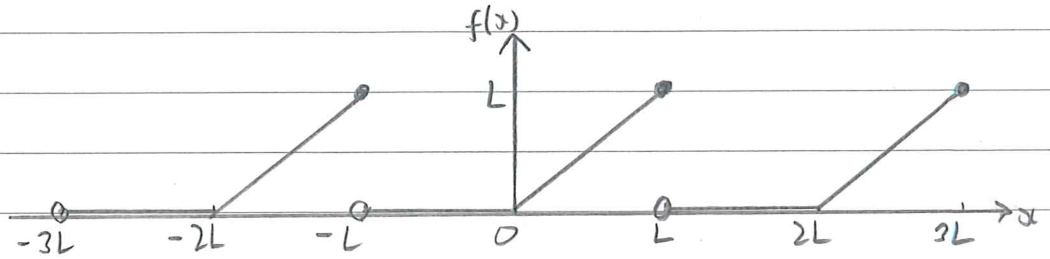
$$\int_{-L}^{L}\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = L\delta_{mn},$$

where $n,m \in \mathbb{N}\setminus\{0\}$ and $\delta_{mn} = \begin{cases} 1 & \text{for } m=n, \\ 0 & \text{for } m\neq n. \end{cases}$

- <u>FCT</u>: Let $f$ be $2L$-periodic with $f$ and $f'$ p.c. on $(-L,L)$. Then $a_n$ and $b_n$ exist, and

$$\frac{1}{2}\left(f(x+) + f(x-)\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right) \text{ for } x\in\mathbb{R}.$$

**Example 2.3**: Find the FS of the $2L$-periodic function $f$ defined by
$$f(x) = \begin{cases} x & \text{for } 0 < x \le L, \\ 0 & \text{for } -L < x \le 0. \end{cases}$$




$$a_n = \frac{1}{L}\int_0^L x\cos\left(\frac{n\pi x}{L}\right)dx, \quad b_n = \frac{1}{L}\int_0^L x\sin\left(\frac{n\pi x}{L}\right)dx$$

Find $a_0 = \frac{1}{L}\frac{L^2}{2} = \frac{L}{2}$, but for $n > 0$ it is a bit quicker to evaluate

$$a_n + i b_n = \frac{1}{L}\int_0^L \underbrace{x}_{u}\ \underbrace{\exp\left(\frac{in\pi x}{L}\right)}_{v'} dx$$

$$= \left[\frac{1}{L}\underbrace{x}_{u}\ \underbrace{\frac{L}{in\pi}\exp\left(\frac{in\pi x}{L}\right)}_{v}\right]_0^L - \frac{1}{L}\int_0^L \underbrace{1}_{u'}\ \underbrace{\frac{L}{in\pi}\exp\left(\frac{in\pi x}{L}\right)}_{v} dx$$

$$= \frac{L}{in\pi}\exp(in\pi) - \left[\frac{1}{L}\left(\frac{L}{in\pi}\right)^2\exp\left(\frac{in\pi x}{L}\right)\right]_0^L$$

$$= \frac{iL(-1)^{n+1}}{n\pi} + \frac{L}{n^2\pi^2}\left((-1)^n - 1\right)$$

$$\Rightarrow f(x) \sim \frac{L}{4} + \sum_{m=1}^{\infty}\left(-\frac{2L}{(2m-1)^2\pi^2}\cos\left(\frac{(2m-1)\pi x}{L}\right) + \frac{L(-1)^{m+1}}{m\pi}\sin\left(\frac{m\pi x}{L}\right)\right)$$ $\square$

- FCT $\Rightarrow$ FS converges to $f(x)$ for $x \neq (2k+1)L$, $k \in \mathbb{Z}$, and to $\frac{1}{2}(f(L+) + f(L-)) = \frac{1}{2}(0 + L) = \frac{L}{2}$ otherwise.
- E.g. $x = 0 \Rightarrow 0 = f(0) = \frac{L}{4} - \frac{2L}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2} \Rightarrow \sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$

  $x = L \Rightarrow \frac{L}{2} = \frac{L}{4} - \frac{2L}{\pi^2}\sum_{m=1}^{\infty}\frac{(-1)}{(2m-1)^2} \Rightarrow$ same sum!

## Cosine and sine series

- Suppose now $f:[0,L]\to\mathbb{R}$ given. Periodic extension of period $2L$ not unique, but there are two especially useful ones for PDE applications.

- **Def$^n$**: The even/odd $2L$-periodic extensions, $f_e$ and $f_o$ respectively, of $f:[0,L]\to\mathbb{R}$ are defined by
$$f_e(x)=\begin{cases} f(x) & \text{for } 0\le x\le L,\\ f(-x) & \text{for } -L<x<0,\end{cases} \quad \text{with } f_e(x+2L)=f_e(x) \text{ for } x\in\mathbb{R}$$
and
$$f_o(x)=\begin{cases} f(x) & \text{for } 0\le x\le L,\\ -f(-x) & \text{for } -L<x<0,\end{cases} \quad \text{with } f_o(x+2L)=f_o(x) \text{ for } x\in\mathbb{R}.$$
*→

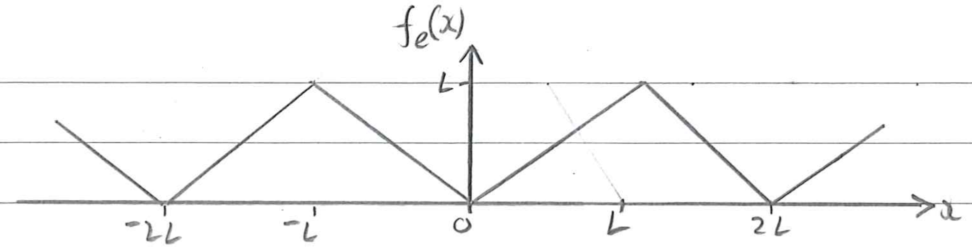
- **Def$^n$**: The Fourier cosine and sine series for $f:[0,L]\to\mathbb{R}$ are the Fourier series for $f_e$ and $f_o$ respectively, i.e.
$$f_e(x)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi x}{L}\right), \text{ where } a_n=\frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx;$$
$$f_o(x)\sim\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi x}{L}\right), \text{ where } b_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

- Note that if $f$ is continuous on $[0,L]$ and $f'$ p.c. on $[0,L]$, then FCT gives
$$\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi x}{L}\right)=f_e(x) \text{ for } x\in\mathbb{R};$$
$$\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi x}{L}\right)=\begin{cases} f_o(x) & \text{for } x\neq kL,\ k\in\mathbb{Z},\\ 0 & \text{otherwise.}\end{cases}$$

- Example 2.4: Find the cosine and sine series of $f:[0,L]\to\mathbb{R}$ defined by $f(x)=x$ for $0\le x\le L$.

$$f_e(x)=\begin{cases} x & \text{for } 0\le x\le L,\\ -x & \text{for } -L<x<0\end{cases}, \text{ i.e. } f_e(x)=|x| \text{ for } -L<x\le L$$

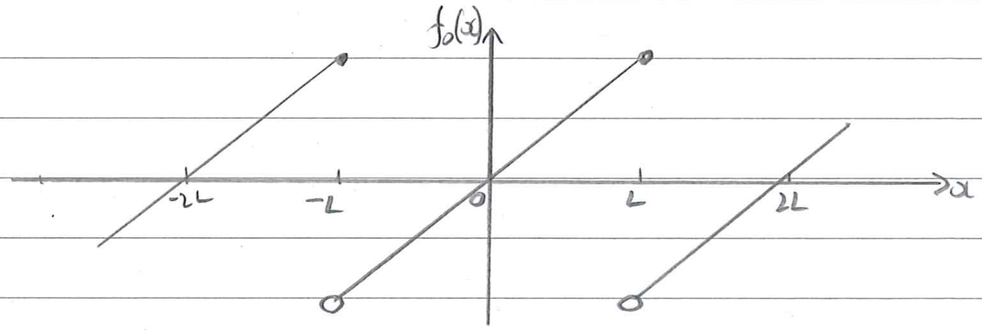
* Note that $f_o(x)$ is odd for $x\neq kL$, $k\in\mathbb{Z}$, and odd (on $\mathbb{R}$) iff $f(0)=f(L)=0$.




$$a_n = \frac{2}{L}\int_0^L x\cos\left(\frac{n\pi x}{L}\right)dx$$

$$\Rightarrow \quad f_e(x) \sim \underbrace{\frac{L}{2} - \sum_{m=0}^{\infty} \frac{4L}{(2m+1)^2\pi^2}\cos\left(\frac{(2m+1)\pi x}{L}\right)}_{\text{Cosine series}} \underset{\text{FCT}}{\overset{\uparrow}{=}} f_e(x)$$

$$f_o(x) = \begin{cases} x & \text{for } 0 \le x \le L, \\ -(-x) & \text{for } -L < x < L \end{cases} \quad \Rightarrow \quad f_o(x) = x \text{ for } -L < x \le L$$




$$b_n = \frac{2}{L}\int_0^L x\sin\left(\frac{n\pi x}{L}\right)dx$$

$$\Rightarrow \quad f_o(x) \sim \underbrace{\sum_{n=1}^{\infty} \frac{2L(-1)^{n+1}}{n\pi}\sin\left(\frac{n\pi x}{L}\right)}_{\text{Sine series}} \underset{\text{FCT}}{\overset{\uparrow}{=}} \begin{cases} f_o(x) & \text{for } x \ne kL,\ k \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

$\square$

## Remarks

(1) $f_e + f_o = 2f_{ex.23} \Rightarrow FS(f_e) + FS(f_o) = FS(2f_{ex.23})$

(2) Rates of convergence? $p=1$ for $f_e$ ✓ $p=0$ for $f_o$ ✓

(3) Qn: Which truncated series gives best approx to $f$ on $[0, L]$?
Ans: Cosine series, since (i) it converges everywhere on $[0, L]$;
(ii) it converges more rapidly;
(iii) it does not exhibit Gibb's phenomenon.