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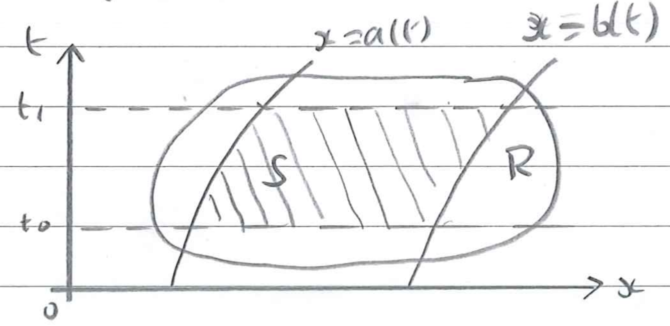
The PDEs we shall study

PDE	Name	Unknown	Parameters
$T_t = \kappa T_{xx}$	Heat equation	$T(x,t)$	$\kappa > 0$
$y_{tt} = c^2 y_{xx}$	Wave equation	$y(x,t)$	$c > 0$
$T_{xx} + T_{yy} = 0$	Laplace's equation	$T(x,y)$	None

- We shall derive them using physical principles and develop methods to solve several physically important problems formed by imposing appropriate BCs and/or ICs — different for each of them!

Some preliminaries

- Leibniz's Integral Rule (LIR)



If F, F_t are continuous on $R \supseteq S$ and $a, \dot{a}, b, \dot{b}$ are continuous for $t \in [t_0, t_1]$ then

$$\frac{d}{dt}\int_{a(t)}^{b(t)} F(x,t)\,dx = \int_{a(t)}^{b(t)} F_t(x,t)\,dx + F(b(t),t)\,\dot{b}(t) - F(a(t),t)\,\dot{a}(t).$$

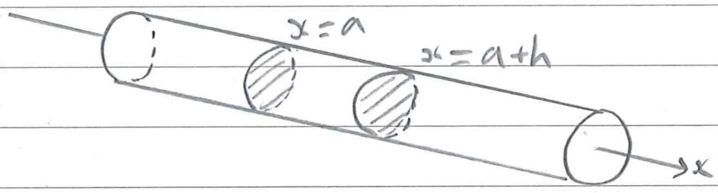
Note: a, b constant $\Rightarrow$ $\frac{d}{dt}\int_a^b F(x,t)\,dx = \int_a^b F_t(x,t)\,dx$

- Lemma (1.2): $f(x)$ cts $\Rightarrow$ $\frac{1}{h}\int_a^{a+h} f(x)\,dx \to f(a)$ as $h \to 0$.

The heat equation

Derivation in 1D

- Consider a straight rigid isotropic conducting rod (e.g. metal) with insulated lateral surfaces lying along x-axis.



- We'll need following quantities.

Symbol	Quantity	SI units
x	Axial distance	m
t	Time	s
$T(x,t)$	Temperature	K
$q(x,t)$	Heat flux in the x-direction	$J m^{-2} s^{-1}$ $(1J = 1Nm)$
A	Cross-sectional area	m^2
ρ	Rod density	$Kg\, m^{-3}$
c	Rod specific heat	$J\, Kg^{-1} K^{-1}$
k	Rod thermal conductivity	$(J\, K^{-1} m^{-1} s^{-1})$
κ	Rod thermal diffusivity	$(m^2 s^{-1})$

- Conservation of energy in fixed section $a \le x \le a+h$:

$$\underbrace{\frac{d}{dt}\left(A\int_a^{a+h} \rho c T \, dx\right)}_{①} = \underbrace{Aq(a,t)}_{②} - \underbrace{Aq(a+h,t)}_{③}.$$

① is time rate of change of internal energy in $a \le x \le a+h$.

② is rate at which heat enters through $x = a$.

③ is rate at which heat leaves through $x = a+h$.

- Note also true for $h<0$ with appropriate reinterpretation.
- Assuming T_t is cts, LIR with $a, a+h$ constant gives

$$\frac{\rho c}{h}\int_a^{a+h} T_t \, dx + \frac{q(a+h,t)-q(a,t)}{h} = 0$$

- Assuming q_x is cts and taking limit $h\to 0$, Lemma(1.2) gives

$$\rho c T_t + q_x = 0. \qquad (+)$$

Fourier's law

- This is the constitutive law

$$q = -k T_x \qquad (\#)$$

- Models flow of heat from high to low temperatures.
- $(+)$ & $(\#)$ $\Rightarrow$ $\rho c T_t - (k T_x)_x = 0$ or $\boxed{T_t = \kappa T_{xx}}$ Heat equation
where $\kappa = k/\rho c$.
- Note we assumed T_t and $q_x = -k T_{xx}$ to be cts.

Units and nondimensionalization

- Notation: $[p]$ = dimension of p in fundamental dimensions (M, L, T, Θ etc) or e.g. SI units (kg, m, s, K etc).
- Both sides of an equation modelling a physical process must have same dimensions, e.g. $[①]=[②]=[③]=Js^{-1}$.
- Exploit to check solutions are dimensionally correct and to determine dimensions of parameters, e.g.

$$[k] = \frac{[q]}{[T_x]} = \frac{J\,m^{-2}s^{-1}}{K\,m^{-1}} = J K^{-1} m^{-1} s^{-1}, \quad \kappa = \frac{[T_t]}{[T_{xx}]} = \frac{[x^2]}{[t]} = m^2 s^{-1}.$$

- **Nondimensionalization**: Method of scaling variables with typical values to derive dimensionless equations. These usually contain dimensionless parameters that characterize the relative importance of the physical mechanisms in the model.

E.g. IBVP

- Suppose $T(x,t)$ s.t. ① $T_t = \kappa T_{xx}$ for $0 < x < L$, $t > 0$;
  ② $T(0,t) = T_0$, $T(L,t) = T_1$ for $t > 0$;
  ③ $T(x,0) = T_2 \frac{x}{L}\left(1 - \frac{x}{L}\right)$ for $0 < x < L$.

- Five dimensional parameters: κ, L, T_0, T_1, T_2.

- Nondimensionalize by scaling $x = L\hat{x}$, $t = L^2\hat{t}/\kappa$ and $T(x,t) = T_2\hat{T}(\hat{x},\hat{t})$, where L^2/κ is the timescale for diffusive transport of heat.

- Chain rule $\Rightarrow$ $\frac{\partial T}{\partial t} = T_2 \frac{\partial \hat{T}}{\partial \hat{t}}\frac{d\hat{t}}{dt} = \frac{\kappa T_2}{L^2}\frac{\partial \hat{T}}{\partial \hat{t}}$

  $\frac{\partial T}{\partial x} = T_2 \frac{\partial \hat{T}}{\partial \hat{x}}\frac{d\hat{x}}{dx} = \frac{T_2}{L}\frac{\partial \hat{T}}{\partial \hat{x}}$ etc

- Hence ① - ③ $\Rightarrow$ dimensionless problem for $\hat{T}(\hat{x},\hat{t})$ given by ①′ $\hat{T}_{\hat{t}} = \hat{T}_{\hat{x}\hat{x}}$ for $0 < \hat{x} < 1$, $\hat{t} > 0$;
  ②′ $\hat{T}(0,\hat{t}) = \alpha_0$, $\hat{T}(1,\hat{t}) = \alpha_1$ for $\hat{t} > 0$;
  ③′ $\hat{T}(\hat{x},0) = \hat{x}(1-\hat{x})$ for $0 < \hat{x} < 1$.

- Two dimensionless parameters $\alpha_0 = \frac{T_0}{T_2}$, $\alpha_1 = \frac{T_1}{T_2}$.

- If $\hat{T} = \hat{T}(\hat{x}, \hat{t}; \alpha_0, \alpha_1)$ is a sol$^{\underline{n}}$ of ①′ - ③′, then a sol$^{\underline{n}}$ of ① - ③ is given by

$$\frac{T}{T_2} = \hat{T}\left(\frac{x}{L}, \frac{\kappa t}{L^2}; \frac{T_0}{T_2}, \frac{T_1}{T_2}\right).$$

i.e. T/T_2 must be a fn of x/L and $\kappa t/L^2$!

Heat conduction in a finite rod

- Consider IBVP for $T(x,t)$ given by
  ① $T_t = \nu_s T_{xx}$ for $0 < x < L$, $t > 0$;
  ② $T(0,t) = 0$, $T(L,t) = 0$ for $t > 0$;
  ③ $T(x,0) = f(x)$ for $0 < x < L$,

Where the initial temperature profile $f(x)$ is given.

- Solve using Fourier's method:

(I) Use method of separation of variables to find the countably infinite set of nontrivial separable sol^ns satisfying the PDE ① and BCs ②, each containing an arbitrary constant.

(II) Use the principle of superposition of a linear problem – that the sum of any number of solutions is also a solution (assuming convergence) – to form the general series solution that is the infinite sum of the sep. sol^ns of PDE & BCs.

(III) Use the theory of Fourier series to determine the constants in the general series solution for which it satisfies the IC ③.

- Remarks:

(1) ① & ② are linear since, if T_1 and T_2 satisfy them, then so too does $\alpha T_1 + \beta T_2$ $\forall \alpha, \beta \in \mathbb{R}$.

(2) To verify resulting series is actually a solution of PDE, need it to converge suff. rapidly that T_t and T_{xx} can be computed by termwise differentiation – we largely gloss over such issues, i.e. we proceed formally.

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Step (I)

- $T = F(x)G(t) \underset{①}{\Rightarrow} FG' = \kappa F''G \Rightarrow \frac{F''}{F} = \frac{G'}{\kappa G}$ $(FG \neq 0)$

- LHS ind. t & RHS ind. x $\Rightarrow$ LHS = RHS ind. x & t
  $\Rightarrow$ LHS = RHS $= -\lambda$, say, $\lambda \in \mathbb{R}$.

Hence, $\underline{\underline{-F''(x) = \lambda F(x) \text{ for } 0 < x < L}}$ $(\dagger)$

- ② $\Rightarrow F(0)G(t) = 0$ and $F(L)G(t) = 0$ for $t > 0$.

T nontrivial $\Rightarrow$ G nontrivial $\Rightarrow$ $\underline{\underline{F(0) = 0, F(L) = 0}}$ $(\ddagger)$

- Now need to find all $\lambda \in \mathbb{R}$ s.t. ODE BVP $(\dagger)-(\ddagger)$ for $F(x)$ has a nontrivial solution. Consider cases.

(i) $\lambda = -w^2$ ($w > 0$ wlog)

$(\dagger) \Rightarrow F'' - w^2 F = 0 \Rightarrow F = A\cosh(wx) + B\sinh(wx)$ $(A, B \in \mathbb{R})$

$(\ddagger) \Rightarrow A = 0, B\sinh(wL) = 0 \Rightarrow F = 0.$

(ii) $\lambda = 0$

$(\dagger) \Rightarrow F'' = 0 \Rightarrow F = A + Bx$ $(A, B \in \mathbb{R})$

$(\ddagger) \Rightarrow A = 0, BL = 0 \Rightarrow F = 0.$

(iii) $\lambda = w^2$ ($w > 0$ wlog)

$(\dagger) \Rightarrow F'' + w^2 F = 0 \Rightarrow F = A\cos(wx) + B\sin(wx)$ $(A, B \in \mathbb{R})$

$(\ddagger) \Rightarrow A = 0, B\sin(wL) = 0$. But $B \neq 0$ for F nontrivial, so $\sin(wL) = 0$, so $wL = n\pi$, $n \in \mathbb{N} \setminus \{0\}$.

- For $\lambda = \omega^2 = \left(\frac{n\pi}{L}\right)^2$, $F = B\sin\left(\frac{n\pi x}{L}\right)$ and $G \propto \exp\left(-\kappa\left(\frac{n\pi}{L}\right)^2 t\right)$
- Hence, nontrivial separable solutions given by

$$T_n(x,t) = b_n \sin\left(\frac{n\pi x}{L}\right)\exp\left(-\frac{n^2\pi^2\kappa t}{L^2}\right),$$

where n is a positive integer and b_n a constant.

Step (II)

- Since ①-② are linear, formally the principal of superposition implies that the general series solution is given by

$$T(x,t) = \sum_{n=1}^{\infty} T_n(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)\exp\left(-\frac{n^2\pi^2\kappa t}{L^2}\right).$$

Step (III)

IC ③ can only be satisfied if

$$f(x) = T(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \text{ for } 0 < x < L.$$

The theory of FS $\Rightarrow$ the Fourier coefficients are given by

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \quad \text{for } n \in \mathbb{N}\setminus\{0\}, \quad (\ddagger)$$

which determines the b_n and hence a solution.

Remarks

(1) f, f' p.c. on $(0,L)$ $\Rightarrow$ sine series converges to $\frac{1}{2}(f(x+)+f(x-))$ for $x \in (0,L)$ and to 0 for $x = 0, L$, so can deal with jump discontinuities in ICs.

(2) In questions often asked to derive (††) via orthogonality relations rather than quoting it. The relevant ones here are

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{L}{2}\delta_{mn},$$

where $m, n \in \mathbb{N}\setminus\{0\}$. Assuming $\int\sum = \sum\int$ then gives, for $n \in \mathbb{N}\setminus\{0\}$,

$$\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_0^L \sum_{m=1}^{\infty} b_m \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \sum_{m=1}^{\infty} b_m \frac{2}{L}\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \sum_{m=1}^{\infty} b_m \delta_{mn}$$

$$= b_n \qquad \square$$

<u>Example 3.1</u>: $f(x) = \sin\left(\frac{n\pi x}{L}\right) + \frac{1}{2}\sin\left(\frac{2\pi x}{L}\right)$

$\Rightarrow b_1 = 1,\ b_2 = \frac{1}{2},\ b_n = 0$ otherwise.

<u>Example 3.2</u>: $f(x) = \begin{cases} T^* & \text{for } L_1 < x < L_2 \\ 0 & \text{otherwise} \end{cases}$

$$\Rightarrow b_n = \frac{2}{L}\int_{L_1}^{L_2} T^* \sin\left(\frac{n\pi x}{L}\right)dx = \frac{2T^*}{n\pi}\left(\cos\left(\frac{n\pi L_1}{L}\right) - \cos\left(\frac{n\pi L_2}{L}\right)\right)$$

- We've found a solution (assuming suff. rapid convergence), but is it the only solution?

<u>Uniqueness</u>

<u>Theorem (3.1)</u>: The IBVP has only one solution.

Pf: Suppose $T, \tilde{T}$ are solutions and let $W = T - \tilde{T}$.