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(2) In questions often asked to derive (##) via orthogonality relations rather than quoting it. The relevant ones here are

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{L}{2}\delta_{mn},$$

where $m, n \in \mathbb{N}\setminus\{0\}$. Assuming $\int\Sigma = \Sigma\int$ then gives, for $n \in \mathbb{N}\setminus\{0\}$,

$$\begin{aligned}\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx &= \frac{2}{L}\int_0^L \sum_{m=1}^{\infty} b_m \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx \\ &= \sum_{m=1}^{\infty} b_m \frac{2}{L}\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx \\ &= \sum_{m=1}^{\infty} b_m \delta_{mn} \\ &= b_n \end{aligned}$$

$\square$

Example 3.1: $f(x) = \sin\left(\frac{n\pi x}{L}\right) + \frac{1}{2}\sin\left(\frac{2\pi x}{L}\right)$

$\Rightarrow b_1 = 1,\ b_2 = \frac{1}{2},\ b_n = 0$ otherwise.

Example 3.2: $f(x) = \begin{cases} T^* & \text{for } L_1 < x < L_2 \\ 0 & \text{otherwise} \end{cases}$

$$\Rightarrow b_n = \frac{2}{L}\int_{L_1}^{L_2} T^* \sin\left(\frac{n\pi x}{L}\right)dx = \frac{2T^*}{n\pi}\left(\cos\left(\frac{n\pi L_1}{L}\right) - \cos\left(\frac{n\pi L_2}{L}\right)\right)$$

- We've found a solution (assuming suff. rapid convergence), but is it the only solution?

Uniqueness

Theorem (3.1): The IBVP has only one solution.

Pf: Suppose $T, \tilde{T}$ are solutions and let $W = T - \tilde{T}$.

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By linearity, ① - ③ $\Rightarrow$

(1') $W_t = T_t - \tilde{T}_t = \kappa T_{xx} - \kappa \tilde{T}_{xx} = \kappa (T - \tilde{T})_{xx} = \kappa W_{xx}$ for $0 < x < L$, $t > 0$;

(2') $W = T - \tilde{T} = 0$ at $x = 0, L$ for $t > 0$;

(3') $W(x,0) = T(x,0) - \tilde{T}(x,0) = f(x) - f(x) = 0$ for $0 < x < L$.

Strategy: deduce that $W(x,t) \equiv 0$.

Trick: analyse $I(t) := \frac{1}{2}\int_0^L W(x,t)^2 \, dx$.

Evidently $I(t) \geq 0$ for $t \geq 0$ and $I(0) = 0$ by (3').

But
$$\begin{aligned}
\frac{dI}{dt} &= \int_0^L W W_t \, dx && \text{(by LIR)} \\
&= \int_0^L W \kappa W_{xx} \, dx && \text{(by (1'))} \\
&= \left[\kappa W W_x\right]_0^L - \kappa \int_0^L W_x W_x \, dx && \text{(by IBP)} \\
&= -\kappa \int_0^L W_x^2 \, dx && \text{(by (2'))} \\
&\leq 0,
\end{aligned}$$

so $I(t)$ cannot increase!

Hence, $0 \leq I(t) \leq I(0) = 0$, giving $I(t) = 0$ for $t \geq 0$, so that $W = 0$ and $T = \tilde{T}$ for $0 \leq x \leq L$, $t \geq 0$ (assuming cty of W there). $\square$

- Note that this method of proof works for any linear BCs for which $[W W_x]_0^L \leq 0$, e.g. the radiative BCs $W_x(0,t) = -\alpha W(0,t)$, $W_x(L,t) = \alpha W(L,t)$ for $t > 0$, where α is a positive parameter.

Non-zero steady state

Example 3.3: Solve the IBVP ① $T_t = \kappa T_{xx}$ for $0 < x < L$, $t > 0$;
② $T(0,t) = T_0$, $T(L,t) = T_1$ for $t > 0$;
③ $T(x,0) = 0$ for $0 < x < L$.
where T_0, T_1 are prescribed constants.

- We cannot use separation of variables straight away because BCs are not homogeneous (unless $T_0 = T_1 = 0$).

- Conjecture that $T(x,t) \to S(x)$ as $t \to \infty$, where $S(x)$ is the steady-state solution of ①-②, so that

$$0 = \kappa S_{xx} \text{ for } 0 < x < L \text{ with } S(0) = T_0,\ S(L) = T_1.$$

- Thus, $S(x) = T_0\left(1 - \frac{x}{L}\right) + T_1\left(\frac{x}{L}\right)$, a linear temp. profile.

- Now let $T(x,t) = S(x) + u(x,t)$, then ①-③ $\Rightarrow u(x,t)$ satisfies the IBVP

①' $(S+u)_t = \kappa(S+u)_{xx} \Rightarrow u_t = \kappa u_{xx}$ for $0 < x < L$, $t > 0$;
②' $S(0) + u(0,t) = T_0$, $S(L) + u(L,t) = T_1 \Rightarrow u(0,t) = 0, u(L,t) = 0$ for $t > 0$;
③' $S(x) + u(x,0) = 0 \Rightarrow u(x,0) = -S(x)$ for $0 < x < L$.

- We solved this problem last lecture using Fourier's method:

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{n^2\pi^2\kappa t}{L^2}\right)$$

where $B_n = -\frac{2}{L}\int_0^1 S(x)\sin\left(\frac{n\pi x}{L}\right)dx = -\frac{2}{n\pi}\left(T_0 - (-1)^n T_1\right)$.

$\square$

- Note that T_0, T_1 in BCs ② end up in IC ③' — sometimes called "method of shifting the data."

Other BCs

Example 3.4: Solve the IBVP ① $T_t = \kappa T_{xx}$ for $0 < x < L$, $t > 0$;
② $T_x(0,t) = 0$, $T_x(L,t) = 0$ for $t > 0$;
③ $T(x,0) = f(x)$ for $0 < x < L$.

- Note both ends thermally insulated since $q = -k T_x = 0$ at $x = 0, L$.

- Apply Fourier's method on problem sheet 4 $\Rightarrow$

$$T(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{n^2\pi^2 \kappa t}{L^2}\right),$$

where $a_n = \frac{2}{L}\int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$.

Remarks

(1) The constant separable solution $T = \frac{a_0}{2}$ of ①-② comes from case in which the separation constant is zero.

(2) As $t \to \infty$, $T(x,t) \to \frac{a_0}{2} = \frac{1}{L}\int_0^L f(x)\,dx$, i.e. mean of initial temp.

(3) Uniqueness by similar argument to before.

Inhomogeneous PDE & BCs

Example 3.5: Solve the IBVP ① $\rho c T_t = k T_{xx} + Q(x,t)$ for $0 < x < L$, $t > 0$;
② $T_x(0,t) = \phi(t)$, $T_x(L,t) = \psi(t)$ for $t > 0$;
③ $T(x,0) = f(x)$ for $0 < x < L$;

where $Q(x,t)$, $\phi(t)$, $\psi(t)$ and $f(x)$ are given.

- Note Q is volumetric heat source (e.g. due to radiation or chemical reaction) and heat flux in positive x-direction $q = -kT_x$.

- Now both PDE and BCs are inhomogeneous!
- Deal first with BCs by shifting the data.
- Find $s(x,t)$ s.t. $s_x(0,t) = \phi(t)$, $s_x(L,t) = \psi(t)$ for $t > 0$, e.g. $s(x,t) = -\phi(t)\frac{(x-L)^2}{2L} + \psi(t)\frac{x^2}{2L}$.
- Let $T(x,t) = s(x,t) + u(x,t)$, then ①-③ $\Rightarrow u(x,t)$ satisfies the IBVP

(1') $\rho c u_t = k u_{xx} + \tilde{Q}(x,t)$ for $0 < x < L$, $t > 0$;
(2') $u_x(0,t) = 0$, $u_x(L,t) = 0$ for $t > 0$;
(3') $u(x,0) = \tilde{f}(x)$ for $0 < x < L$;

where
$$\left.\begin{aligned} \tilde{Q}(x,t) &= Q(x,t) + k s_{xx} - \rho c s_t \\ \tilde{f}(x) &= f(x) - s(x,0) \end{aligned}\right\} \text{Known in terms of } Q, \phi, \psi, f.$$

- If $\tilde{Q} = 0$, then can solve (1')-(3') via Fourier's method as in Example 3.4.
- This suggests we seek a solution of the form

$$u(x,t) = \frac{u_0(t)}{2} + \sum_{n=1}^{\infty} u_n(t) \cos\left(\frac{n\pi x}{L}\right), \qquad (+)$$

where the functions $u_0(t), u_1(t), \ldots$ are TBD.

- Since (+) is a Fourier cosine series, its Fourier coefficients are given by

$$u_n(t) = \frac{2}{L}\int_0^L u(x,t)\cos\left(\frac{n\pi x}{L}\right) dx \quad \text{for } n \in \mathbb{N};$$

- We can then use (1')-(3') to derive ODEs for the u_n's, as follows.

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- By Leibniz's integral rule,

$$\rho c \frac{dU_n}{dt} = \frac{2}{L}\int_0^L \rho c U_t \cos\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{2}{L}\int_0^L (kU_{xx} + \tilde{Q})\cos\left(\frac{n\pi x}{L}\right) dx \qquad \text{(by (1'))}$$

$$= \frac{2k}{L}\int_0^L U_{xx}\cos\left(\frac{n\pi x}{L}\right) dx + \tilde{Q}_n(t),$$

where $\tilde{Q}_n(t) = \frac{2}{L}\int_0^L \tilde{Q}(x,t)\cos\left(\frac{n\pi x}{L}\right) dx$ are the known coefficients of the Fourier cosine series for $\tilde{Q}$.

- How do we deal with the U_{xx} integral? IBP twice via

$$(uv' - u'v)' = uv'' - u''v \Rightarrow [uv' - u'v]_a^b = \int_a^b uv'' - u''v \, dx.$$

Let $u = U$, $v = \cos\left(\frac{n\pi x}{L}\right)$, $a = 0$, $b = L$, then

$$\underbrace{\left[U\left(-\frac{n\pi}{L}\right)\sin\left(\frac{n\pi x}{L}\right) - U_x \cos\left(\frac{n\pi x}{L}\right)\right]_0^L}_{=0 \text{ by } (2')} = \int_0^L U\left(-\frac{n^2\pi^2}{L^2}\cos\left(\frac{n\pi x}{L}\right)\right) - U_{xx}\cos\left(\frac{n\pi x}{L}\right) dx$$

$$\Rightarrow \frac{2}{L}\int_0^L U_{xx}\cos\left(\frac{n\pi x}{L}\right) dx = -\frac{n^2\pi^2}{L^2}\frac{2}{L}\int_0^L U\cos\left(\frac{n\pi x}{L}\right) dx = -\frac{n^2\pi^2}{L^2} U_n.$$

Hence, $\rho c \frac{dU_n}{dt} + \frac{kn^2\pi^2}{L^2} U_n = \tilde{Q}_n(t)$ for $t > 0$.

- IC? (3') $\Rightarrow$ $U_n(0) = \frac{2}{L}\int_0^L \tilde{f}(x)\cos\left(\frac{n\pi x}{L}\right) dx$

Remarks

(1) Reduced problem to a countably infinite set of ODEs — recover solution of Example 3.4 when $Q = 0, \phi = 0, \psi = 0$.

(2) Can solve explicitly for the U_n's using an integrating factor.

(3) Uniqueness proof the same as for Example 3.4.

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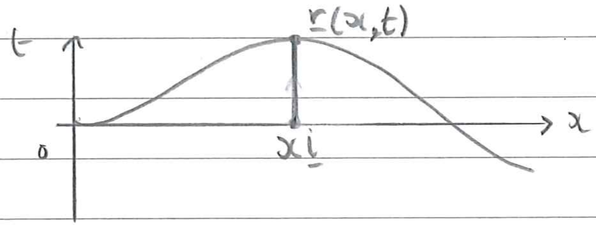
The wave equation

Derivation in 1D

- Consider the small transverse vibrations of a homogeneous extensible elastic string stretched initially along the x-axis at time $t=0$.

- A point at $x\underline{i}$ at time $t=0$ is displaced to $\underline{r}(x,t) = x\underline{i} + y(x,t)\underline{j}$ at time $t>0$, where the transverse displacement $y(x,t)$ is TBD.

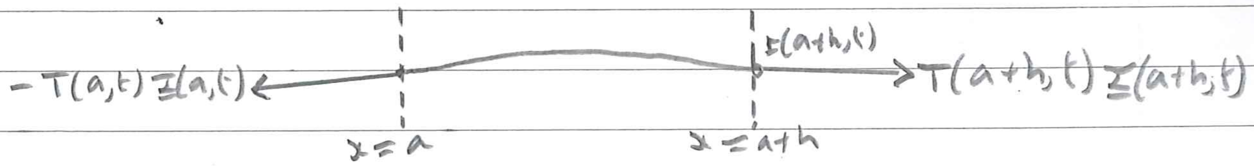


- Consider piece of string in fixed region $a \le x \le a+h$.

- Linear momentum is $\int_a^{a+h} \rho \underline{r}_t \, dx$, where ρ is constant line density of the string ($[\rho] = \text{kg m}^{-1}$)

- Assuming no resistance to bending (cf a ruler), the string to right of $\underline{r}(x,t)$ exerts at this point a force $T(x,t)\underline{I}(x,t)$ on string to left, where $T(x,t)$ is tension ($[T] = N = \text{kg m s}^{-2}$) and $\underline{I} = \underline{r}_x / |\underline{r}_x|$ is unit tangent vector in +ve x-direction.

- Assuming tension so large that gravity and air resistance are negligible, the forces on the string in $a \le x \le a+h$ are:



- NII says $\frac{d}{dt}$(linear momentum) = net force, so

$$\frac{d}{dt}\left(\int_a^{a+h} \rho \underline{r}_t \, dx\right) = T(a+h,t)\underline{I}(a+h,t) - T(a,t)\underline{I}(a,t).$$

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- Assuming $\underline{r}_{tt}$ is cts, LIR then gives

$$\frac{1}{h}\int_a^{a+h} \rho \underline{r}_{tt}\, dx = \frac{T(a+h,t)\,\underline{I}(a+h,t) - T(a,t)\,\underline{I}(a,t)}{h}.$$

- Assuming $(T\underline{I})_x$ is cts, let $h \to 0$ (from above & below) $\Rightarrow$

$$\rho \underline{r}_{tt} = \frac{\partial}{\partial x}(T\underline{I})$$

$$\Rightarrow \quad \rho y_{tt}\underline{j} = \frac{\partial}{\partial x}\left(\frac{T\underline{i} + Ty_x\underline{j}}{(1+y_x^2)^{1/2}}\right)$$

- Now small displacement $\Rightarrow$ small slope $\Rightarrow$ $|y_x| \ll 1$

$$\Rightarrow \quad (1+y_x^2)^{1/2} = 1 + \tfrac{1}{2}(y_x)^2 + \ldots$$

$\Rightarrow$ to a first approximation (i.e. neglecting quadratic e h.o.t.)

$$\rho y_{tt} = T_x\underline{i} + (Ty_x)\underline{j}$$

- x-direction $\Rightarrow T_x = 0 \Rightarrow T = T(t)$, i.e. tension is spacially uniform, but could vary with t, e.g. tuning a guitar string. We shall assume T = constant, which is the case in many practical applications.

- y-direction $\Rightarrow \rho y_{tt} = (Ty_x)_x = Ty_{xx}$

- We have derived the wave equation

$$y_{tt} = c^2 y_{xx},$$

where $c = \sqrt{T/\rho}$ is the wave speed (for reasons that will become apparent).

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Units and nondimensionalization

- $[c^2] = \frac{[y_{tt}]}{[y_{xx}]} = \frac{ms^{-2}}{mm^{-2}} = m^2 s^{-2} \Rightarrow [c] = ms^{-1}.$

Check: $[c^2] = \frac{[T]}{[\rho]} = \frac{N}{Kgm^{-1}} = \frac{Kgms^{-2}}{Kgm^{-1}} = m^2 s^{-2}.$ ✓

- On what timescale does a displacement travel a distance L?

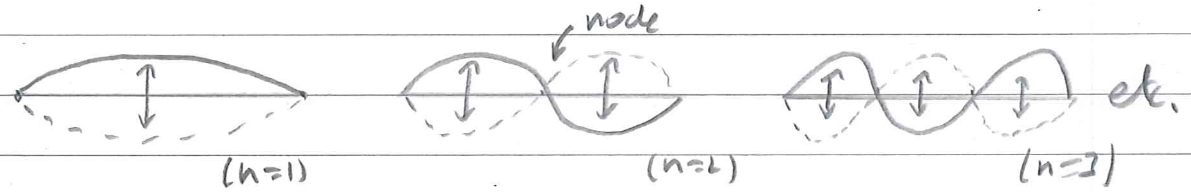
Scale $x = L\hat{x}$, $t = t_0 \hat{t}$, $y = H\hat{y}$

$\Rightarrow \frac{H}{t_0^2} \hat{y}_{\hat{t}\hat{t}} = \frac{Hc^2}{L^2} \hat{y}_{\hat{x}\hat{x}} \Rightarrow \hat{y}_{\hat{t}\hat{t}} = \hat{y}_{\hat{x}\hat{x}}$ provided $t_0 = \frac{L}{c}$.

Normal modes of vibration for a finite string

- Suppose string stretched between $x=0$ and $x=L$ and the ends held fixed.

- Slinky experiment suggest $\exists$ discrete modes of vibration:



- To analyse mathematically we seek separable solutions to ① $y_{tt} = c^2 y_{xx}$ for $0 < x < L$, $t \in \mathbb{R}$;
  ② $y(0,t) = 0$, $y(L,t) = 0$, $t \in \mathbb{R}$.

- $y = F(x)G(t)$ in ① $\Rightarrow FG'' = c^2 F''G \Rightarrow \frac{F''(x)}{F(x)} = \frac{G''(t)}{c^2 G(t)}$ $(FG \neq 0)$

- LHS ind. t & RHS ind. x $\Rightarrow$ LHS = RHS ind. x & t
  $\Rightarrow$ LHS = RHS $= -\lambda \in \mathbb{R}$, say.

- Hence $-F'' = \lambda F$ for $0 < x < L$ (I)

- ② & G non-trivial $\Rightarrow$ $F(0) = 0$, $F(L) = 0$ (II)

- $\lambda \leq 0 \Rightarrow$ (I) - (II) have only the trivial solution $F = 0$.
- Let $\lambda = \omega^2$, with $\omega > 0$ wlog.
- (I) $\Rightarrow$ $F = A\cos(\omega x) + B\sin(\omega x)$ $(A, B \in \mathbb{R})$
- (II) $\Rightarrow$ $A = 0$, $B\sin(\omega L) = 0$
- F nontrivial $\Rightarrow B \neq 0 \Rightarrow \sin(\omega L) = 0 \Rightarrow \omega L = n\pi, n \in \mathbb{N} \setminus \{0\}$.
- $\omega = \frac{n\pi}{L} \Rightarrow F(x) = B\sin\left(\frac{n\pi x}{L}\right), G(t) = C\cos\left(\frac{n\pi c t}{L}\right) + D\sin\left(\frac{n\pi c t}{L}\right)$, $(C, D \in \mathbb{R})$
- Combo $\Rightarrow$ normal modes (nontrivial sep. soln of ① - ②) are

$$y_n(x,t) = \sin\left(\frac{n\pi x}{L}\right)\left(a_n \cos\left(\frac{n\pi c t}{L}\right) + b_n \sin\left(\frac{n\pi c t}{L}\right)\right)$$

or $$y_n(x,t) = c_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi c}{L}(t + \varepsilon_n)\right),$$

where $a_n, b_n \in \mathbb{R}$, $c_n, \varepsilon_n \in \mathbb{R}$ for each $n \in \mathbb{N} \setminus \{0\}$.

Remarks

(1) y_n periodic in t with prime period $p = \frac{2\pi}{n\pi c/L} = \frac{2L}{nc}$ and frequency (or pitch) $\frac{1}{p} = \frac{nc}{2L}$.

(2) y_1 is fundamental mode; $\frac{c}{2L}$ the fundamental frequency; all other modes have a frequency that is an integer multiple of $\frac{c}{2L}$.

(3) Consistent with slinky experiment.

(4) Normal modes are an example of a standing wave since $y = f^n(x) \times$ oscillatory function.

[Next time: use Fourier's method to solve IBVP obtained by imposing 2 ICs.]