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IBVP for a finite string

- Find $y(x,t)$ s.t. ① $y_{tt} = c^2 y_{xx}$ for $0 < x < L, t > 0$;
  ② $y(0,t) = 0$, $y(L,t) = 0$ for $t > 0$;
  ③ $y(x,0) = f(x)$, $y_t(x,0) = g(x)$ for $0 < x < L$.

- Use Fourier's method. [f & g = initial transverse displacement & velocity]

Step (I): Find all nontrivial sep. sol$^{\underline{ns}}$ of ①-②

- Last time we found that these (normal modes) are

$$y_n(x,t) = \sin\left(\frac{n\pi x}{L}\right)\left(a_n \cos\left(\frac{n\pi c t}{L}\right) + b_n \sin\left(\frac{n\pi c t}{L}\right)\right),$$

where $a_n, b_n \in \mathbb{R}$ and $n \in \mathbb{N}\setminus\{0\}$.

Step (II): Formally apply the principle of superposition

- ① & ② linear so superimpose the normal modes (assuming convergence) to obtain the general series solution

$$y(x,t) = \sum_{n=1}^{\infty} y_n(x,t).$$

Step (III): Use theory of FS to satisfy the ICs

- ③ can only be satisfied if

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{for } 0 < x < L,$$

$$g(x) = \sum_{n=1}^{\infty} b_n \frac{n\pi c}{L} \sin\left(\frac{n\pi x}{L}\right) \quad \text{for } 0 < x < L.$$

- Assuming $\int \sum = \sum \int$, we deduce that

$$\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx = \sum_{n=1}^{\infty} a_n \underbrace{\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx}_{=\frac{L}{2}\delta_{nm}} = \frac{L}{2}a_m$$

$$\Rightarrow \quad a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \text{ and similarly}$$

$$b_n\frac{n\pi c}{L} = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$ □

Example 4·1: $f(x) = A\sin\left(\frac{\pi x}{L}\right) + B\sin\left(\frac{2\pi x}{L}\right) \Rightarrow a_1 = A,\ a_2 = B$ and rest 0.

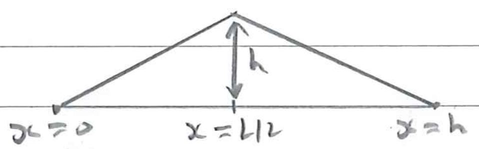
Example 4·2: $f(x)=0,\ g(x)=V_0\sin^3\left(\frac{\pi x}{L}\right) \Rightarrow a_n = 0\ \forall n.$ □

Trick: $\sin^3\left(\frac{\pi x}{L}\right) = \frac{3}{4}\sin\left(\frac{\pi x}{L}\right) - \frac{1}{4}\sin\left(\frac{3\pi x}{L}\right)$

$$\Rightarrow \frac{\pi c}{L}b_1 = \frac{3V_0}{4},\ b_2 = 0,\ \frac{3\pi c}{L}b_3 = -\frac{V_0}{4} \text{ and rest } 0.$$ □

Example 4.3 (Guitar string):

$$f(x) = \begin{cases} 2hx/L & \text{for } 0 \le x \le L/2, \\ 2h(L-x)/L & \text{for } L/2 \le x \le L, \end{cases} \qquad g(x) = 0.$$



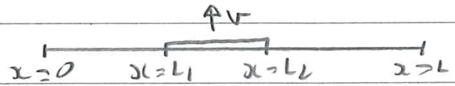
$$a_n = \frac{2}{L}\int_0^{L/2}\frac{2hx}{L}\sin\left(\frac{n\pi x}{L}\right)dx + \frac{2}{L}\int_{L/2}^{L}\frac{2h(L-x)}{L}\sin\left(\frac{n\pi x}{L}\right)dx = \frac{8h}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

$$b_n = 0$$

Since $\sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0 & \text{for } n=2m,\ m\in\mathbb{N}\setminus\{0\}, \\ (-1)^m & \text{for } n=2m+1,\ m\in\mathbb{N}, \end{cases}$

we find $$y(x,t) = \frac{8h}{\pi^2}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^2}\sin\left(\frac{(2m+1)\pi x}{L}\right)\cos\left(\frac{(2m+1)\pi c t}{L}\right)$$ □

Example 4.4 (Piano string):



$$f(x) = 0,\ g(x) = \begin{cases} v & \text{for } L_1 \le L \le L_2, \\ 0 & \text{otherwise.} \end{cases}$$

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$a_n = 0$ and $b_n = \frac{L}{n\pi c}\frac{2}{L}\int_{L_1}^{L_2} v\sin\left(\frac{n\pi x}{L}\right)$

$$\Rightarrow y(x,t) = \frac{2vL}{c\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\cos\left(\frac{n\pi L_2}{L}\right)-\cos\left(\frac{n\pi L_1}{L_1}\right)\right)\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right) \quad \square$$

Energy and uniqueness

- Consider IBVP ①-③.
- KE of string is $\int_0^L \frac{1}{2}\rho|\underline{r}_t|^2\,dx = \int_0^L \frac{1}{2}\rho y_t^2\,dx$.
- Elastic PE of string is product of tension and extension, i.e.

$$T\left(\int_0^L |\underline{r}_x|\,dx - L\right) = T\int_0^L (1+y_x^2)^{1/2}-1\,dx.$$

- But $|y_x| \ll 1$, so $(1+y_x^2)^{1/2}-1 = \frac{1}{2}y_x^2+\dots$, so to a first approximation (neglecting cubic and h.o.t.), the elastic PE is $\int_0^L \frac{1}{2}Ty_x^2\,dx$.
- Hence, energy of string $E(t) = \int_0^L \frac{1}{2}\rho y_t^2 + \frac{1}{2}Ty_x^2\,dx$.

<u>Lemma (4.1)</u>: If y satisfies ①-②, then $E(t)$ is constant for $t>0$.

Pf: $\frac{dE}{dt} = \int_0^L \rho y_t y_{tt} + Ty_x y_{xt}\,dx$ (by LIR)

$= \int_0^L Ty_t y_{xx} + Ty_x y_{xt}\,dx$ (by ①)

$= \int_0^L (Ty_t y_x)_x\,dx$

$= \left[Ty_t y_x\right]_{x=0}^{x=L}$

$= 0,$

since ② $\Rightarrow y_t(0,t) = y_t(L,t) = 0$ for $t>0$. $\square$

Remarks

(1) Lemma & ③ $\Rightarrow$ $E(t) = E(0) = \int_0^L \frac{1}{2}\rho(g(x))^2 + \frac{1}{2}T(f'(x))^2\,dx$.

(2) Lemma $\Rightarrow$ Energy in nth normal mode (given by)

$$E_n(t) = E_n(0) = \int_0^L \frac{1}{2}\rho(y_{nt}(x,0))^2 + \frac{1}{2}T(y_{nx}(x,0))^2\,dx = \frac{n^2\pi^2 T}{4L}(a_n^2+b_n^2).$$

(3) Can then use Parseval's Identity for g and f' to show that $E(0) = \sum_{n=1}^{\infty} E_n(0)$, i.e. total energy is sum of energy in each normal mode (which are constant throughout motion and set by ICs (by remark (2)).

<u>Theorem (4.2, Uniqueness)</u>: The IBVP has at most one solution.

Pf: Let $W(x,t) = y - \tilde{y}$, where $y, \tilde{y}$ are two solutions, then by linearity,

1' $W_{tt} = c^2 W_{xx}$ for $0 < x < L$, $t > 0$;
2' $W(0,t) = 0$, $W(L,t) = 0$ for $t > 0$;
3' $W(0,t) = 0$, $W_t(0,t) = 0$ for $0 < x < L$.

Lemma (4.1) applied to W $\Rightarrow$

$$\int_0^L \frac{\rho}{2}(W_t)^2 + \frac{T}{2}(W_x)^2\,dx = E(t) \underset{1'\,2'}{=} E(0) \underset{3'}{=} 0 \quad \text{for } t \geq 0$$

$\Rightarrow$ $W_t = W_x = 0$ for $0 < x < L$, $t > 0$ (assuming W_t, W_x cts there)

$\Rightarrow$ $W =$ constant for $0 < x < L$, $t > 0$

$\Rightarrow$ $W = 0$ for $0 \leq x \leq L$, $t \geq 0$ (by 2' or 3', assuming W cts for $0 \leq x \leq L$, $t \geq 0$)

$\square$

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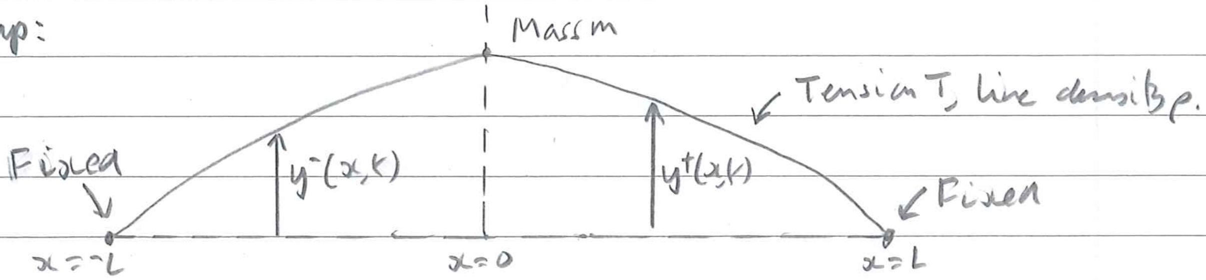
Normal modes for a weighted string

- Setup:



- What are the normal modes?

- PDEs: (1⁻) $y^-_{tt} = c^2 y^-_{xx}$ for $-L < x < 0$
  (1⁺) $y^+_{tt} = c^2 y^+_{xx}$ for $0 < x < L$

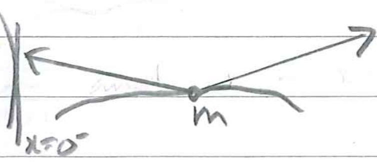
- BCs: (2⁻) $y^-(-L, t) = 0$
  (2⁺) $y^+(L, t = 0$

  (3) $y^-(0,t) = y^+(0,t) = Y(t)$ say.

- $Y(t)$ TBD so need a second BC at $x = 0$ via NII for the mass.

- Forces on mass (neglecting gravity and air resistance):

$$-T\left(\frac{\underline{i} + y^-_x \underline{j}}{(1+(y^-_x)^2)^{1/2}}\right)\Bigg|_{x=0^-} \qquad T\left(\frac{\underline{i} + y^+_x \underline{j}}{(1+(y^+_x)^2)^{1/2}}\right)\Bigg|_{x=0^+}$$



- Small transverse displacement $\Rightarrow |y^\pm_x| \ll 1 \Rightarrow$ $(1+(y^\pm_x)^2)^{1/2} = 1 + \text{h.o.t.} \Rightarrow$ to a first approximation mass remains on y-axis (because x-force components balance), while in y-direction

$$④ \quad m\ddot{Y} = T\left(y^+_x|_{x=0^+} - y^-_x|_{x=0^-}\right)$$

- Separate variables: $y^\pm = F_\pm(x)\,G(t)$

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$(1^\pm) \Rightarrow \dfrac{F_\pm''(x)}{F_\pm(x)} = \dfrac{G''(t)}{c^2 G} = -\lambda \in \mathbb{R}$ say $(F^\pm G \neq 0)$

$(2^\pm)$ & $G \neq 0 \Rightarrow F_-(-L) = 0, \; F_+(L) = 0$ $(a^\pm)$

(3) & $G \neq 0 \Rightarrow F_-(0) = F_+(0)$ (b)

(4) & $G \neq 0 \Rightarrow m F_\pm(0) G''(t) = T(F_+'(0_+) - F_-'(0_-))G(t)$

$\underset{(c^2 = T/\rho)}{\Rightarrow} \; -\dfrac{\lambda m}{\rho} F_\pm(0) = F_+'(0_+) - F_-'(0_-)$ (c)

- Can show $\lambda \leq 0 \Rightarrow F_\pm = 0$. Let $\lambda = \omega^2$, $\omega > 0$ wlog.

- Then $F_-'' + \omega^2 F_- = 0$ for $-L < x < 0$,
  $F_+'' + \omega^2 F_+ = 0$ for $0 < x < L$.

$(a^\pm) \Rightarrow F_- = A \sin \omega(L+x), \; F_+ = B \sin \omega(L-x)$ $(A, B \in \mathbb{R})$

(b) & $(c) \Rightarrow \underbrace{\begin{bmatrix} \sin(\omega L) & -\sin(\omega L) \\ \cos(\omega L) - \frac{m\omega}{\rho}\sin(\omega L) & \cos(\omega L) \end{bmatrix}}_{M} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $(\dagger)$

- $\begin{bmatrix} A \\ B \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \det(M) = 0 \Rightarrow \sin(\omega L)\left(2\cos(\omega L) - \frac{m\omega}{\rho}\sin(\omega L)\right) = 0$

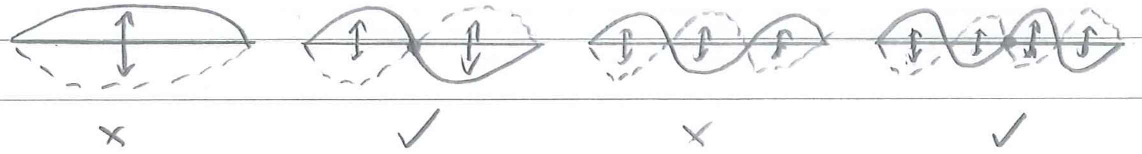
- Hence, either (i) $\sin(\omega L) = 0 \Rightarrow \omega = \dfrac{n\pi}{L}$, $n \in \mathbb{N} \setminus \{0\}$

  or (ii) $\cot(\omega L) = \dfrac{m\omega}{2\rho} \Rightarrow \cot\theta = \dfrac{m\theta}{2\rho L}$, where $\theta = \omega L$

- In each case, $G'' + \omega^2 c^2 G = 0 \Rightarrow G(t) = \underset{1 \text{ wlog}}{\underbrace{C}} \cos(\omega c t + \varepsilon)$ $(C, \varepsilon \in \mathbb{R})$

- In case (i), $(\dagger) \Rightarrow A = -B \Rightarrow \begin{cases} y_- = A \sin\omega(L+x)\cos(\omega c t + \varepsilon) \\ y_+ = -A \sin\omega(L-x)\cos(\omega c t + \varepsilon) \end{cases}$

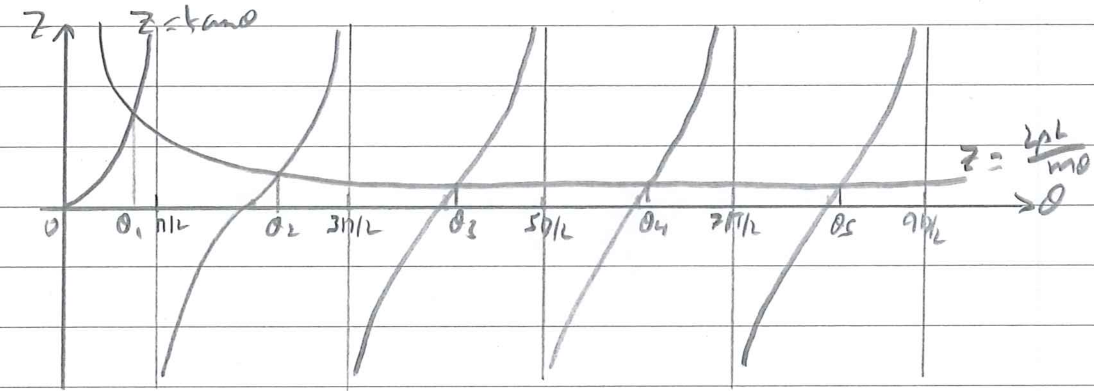
This means that the normal modes are the same as for a string of length $2L$ with a node at $x = 0$, i.e. mass stationary.



etc

- In case (ii) there are infinitely many roots $\theta_1 < \theta_2 < \theta_3 < \dots$

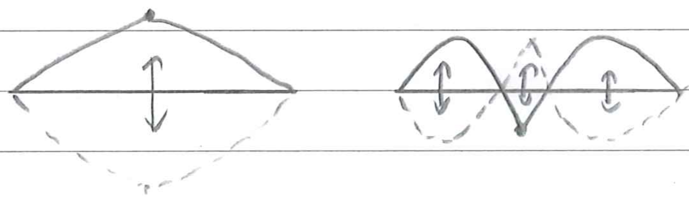
To see plot LHS & RHS of e.g. $\tan\theta = \frac{2\rho L}{m\theta}$



Hence, ∞ many normal modes $\omega_n = \frac{\theta_n}{L}$, $n \in \mathbb{N}\setminus\{0\}$.

- Now $(\dagger) \Rightarrow A = B \Rightarrow \begin{cases} y_- = A\sin\omega(L+x)\cos(\omega c t + \varepsilon) \\ y_+ = A\sin\omega(L-x)\cos(\omega c t+\varepsilon) \end{cases}$

This means that the normal modes are symmetric about $x=0$:



etc

- Try with slinky!

<u>General solution of the wave equation</u>

- Remarkable fact: can write down all solutions of $y_{tt} = c^2 y_{xx}$!

- Let $y(x,t) = Y(\xi,\eta)$, $\xi = x - ct$, $\eta = x + ct$ (as in Intro. Calculus)