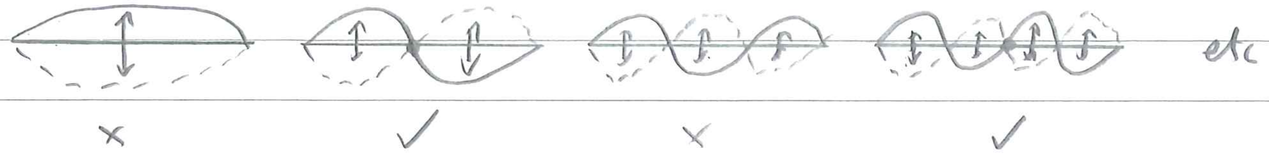
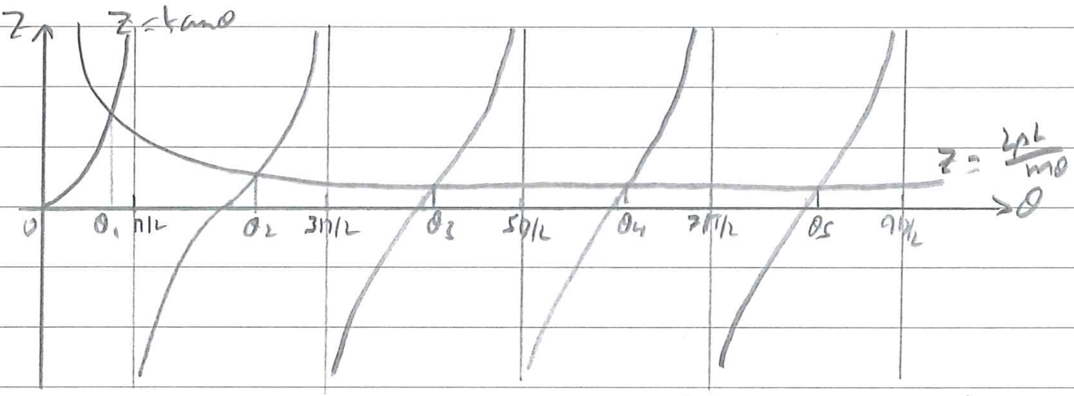


- In case (ii) there are infinitely many roots $\theta_1 < \theta_2 < \theta_3 < \dots$

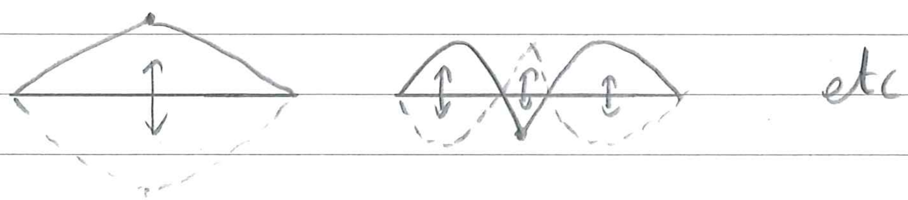
To see plot LHS & RHS of e.g. $\tan\theta = \frac{2\rho L}{m\theta}$



Hence, ∞ many normal modes $\omega_n = \frac{\theta_n}{L}$, $n \in \mathbb{N} \setminus \{0\}$.

- Now (†) $\Rightarrow A = B \Rightarrow \begin{cases} y_- = A \sin\omega(L+x)\cos(\omega c t + \varepsilon) \\ y_+ = A \sin\omega(L-x)\cos(\omega c t + \varepsilon) \end{cases}$

This means that the normal modes are symmetric about $x = 0$:



- Try with slinky!

General solution of the wave equation

- Remarkable fact: can write down all solutions of $y_{tt} = c^2 y_{xx}$!

- Let $y(x,t) = Y(\xi,\eta)$, $\xi = x - ct$, $\eta = x + ct$ (as in Intro. Calculus)

$$\Rightarrow \quad y_x = Y_\xi \xi_x + Y_\eta \eta_x = Y_\xi + Y_\eta$$
$$y_{xx} = (Y_\xi + Y_\eta)_\xi \xi_x + (Y_\xi + Y_\eta)_\eta \eta_x = Y_{\xi\xi} + 2Y_{\xi\eta} + Y_{\eta\eta}$$
$$y_t = Y_\xi \xi_t + Y_\eta \eta_t = -cY_\xi + cY_\eta$$
$$y_{tt} = (-cY_\xi + cY_\eta)_\xi \xi_t + (-cY_\xi + cY_\eta)_\eta \eta_t = c^2(Y_{\xi\xi} - 2Y_{\xi\eta} + Y_{\eta\eta})$$

where we assumed $Y_{\xi\eta} = Y_{\eta\xi}$.

Hence, $y_{tt} - c^2 y_{xx} = -4c^2 Y_{\xi\eta}$

Wave equation, $c > 0 \Rightarrow Y_{\xi\eta} = 0$

$$\Rightarrow Y_\xi = F'(\xi) \text{ say}$$
$$\Rightarrow (Y - F(\xi))_\xi = 0$$
$$\Rightarrow Y - F(\xi) = G(\eta) \text{ say}$$
$$\Rightarrow y = F(x - ct) + G(x + ct)$$

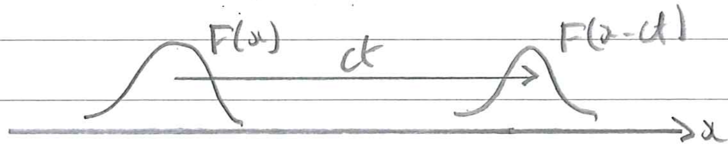
where F, G are arbitrary twice ctsly diff. functions.

Remarks

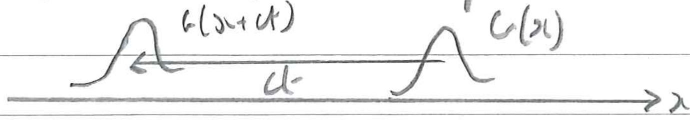
(1) cf. # arb. constants in general solution to a 2^{nd} order ODE.

(2) Easy to verify this is a solution (see online notes): we've shown that all solutions must be of this form.

(4) $F(x-ct)$ is a (travelling) wave of constant shape moving to the right with speed c:



$G(x+ct)$ is a (travelling) wave of constant shape moving to the left with speed c:



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Waves on infinite strings: D'Alembert's formula

- Consider the IVP ① $y_{tt} = c^2 y_{xx}$ for $-\infty < x < \infty$, $t > 0$;
  ② $y(x,0) = f(x)$, $y_t(x,0) = g(x)$ for $-\infty < x < \infty$,

where initial transverse displacement f and velocity g are given.

- The general solution of ① is $y(x,t) = F(x-ct) + G(x+ct)$.

- ICs ② $\Rightarrow$ $F(x) + G(x) \overset{ⓐ}{=} f(x)$, $-cF'(x) + cG'(x) = g(x)$ for $x \in \mathbb{R}$.

The latter implies $-F(x) + G(x) \overset{ⓑ}{=} \frac{1}{c}\int_0^x g(s)\,ds + a$ $(a \in \mathbb{R})$

ⓐ - ⓑ $\Rightarrow$ $F(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(s)\,ds - \frac{1}{2}a$

ⓐ + ⓑ $\Rightarrow$ $G(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(s)\,ds + \frac{1}{2}a$

Hence, $y(x,t) = \frac{1}{2}f(x-ct) + \frac{1}{2c}\int_{x-ct}^0 g(s)\,ds - \frac{1}{2}a$

$$+ \frac{1}{2}f(x+ct) + \frac{1}{2c}\int_0^{x+ct} g(s)\,ds + \frac{1}{2}a$$

$$\Rightarrow \boxed{y(x,t) = \frac{1}{2}\left(f(x-ct) + f(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds}$$

D'Alembert's formula

Remarks

(1) Don't forget the constant a!

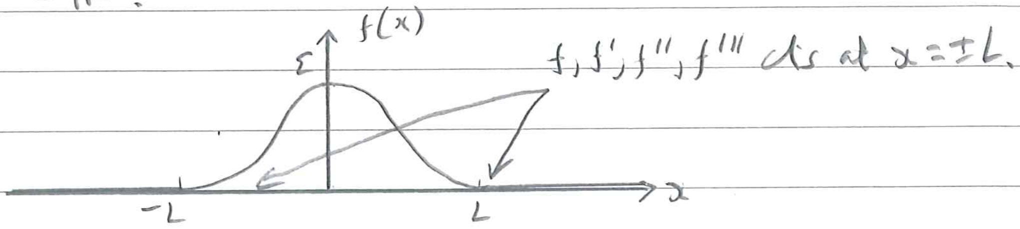
(2) Argument shows $\exists!$ soln of IVP ①-②.

(3) Can also prove uniqueness via energy conservation under the additional assumption that $y_t, y_x \to 0$ suff. rapidly as $x \to \pm\infty$ that the energy $E(t) = \int_{-\infty}^{\infty} \frac{\rho}{2}y_t^2 + \frac{T}{2}y_x^2\,dx$ exists.

(42)

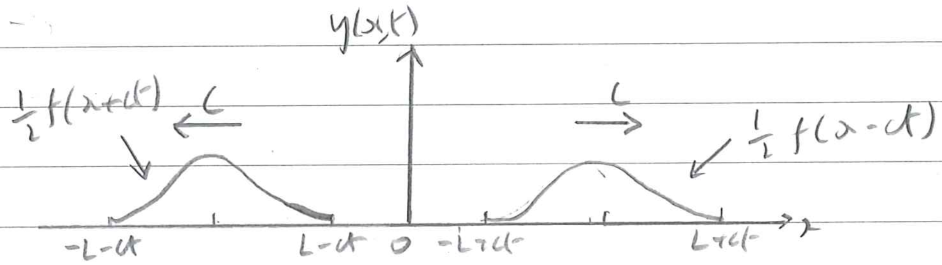
Example: $f(x) = \begin{cases} \varepsilon \cos^4\left(\frac{\pi x}{2L}\right) & \text{for } |x| \le L, \\ 0 & \text{otherwise}, \end{cases} \quad g(x) = 0,$

where $\varepsilon, L \in \mathbb{R}^+$.

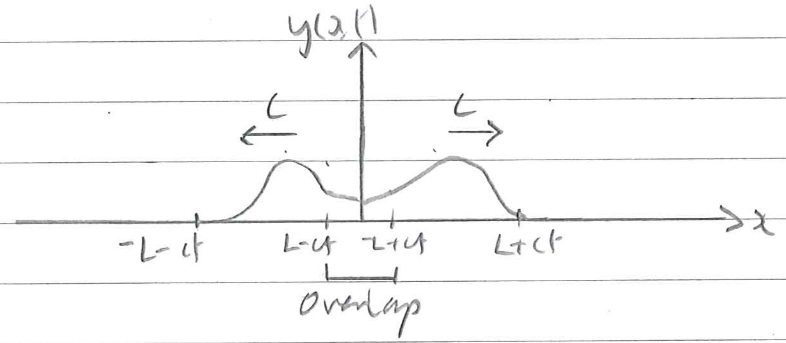


DF $\Rightarrow$ $y(x,t) = \frac{1}{2} f(x-ct) + \frac{1}{2} f(x+ct)$

$ct > L$



$0 < ct < L$



Explicit formulae for these graphs requires some careful book keeping - much easier to use a...

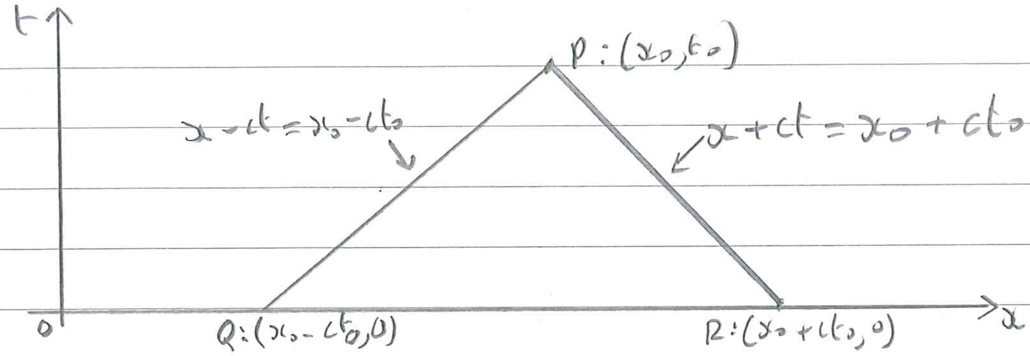
Characteristic diagram

- Let $P = (x_0, t_0) \in \mathbb{R} \times \mathbb{R}^+$. How does $y(P)$ depend on f, g?
- DF $\Rightarrow$ $y(x_0,t_0) = \frac{1}{2}\left(f(x_0 - ct_0) + \frac{1}{2} f(x_0 + ct_0)\right) + \frac{1}{2c}\int_{x_0-ct_0}^{x_0+ct_0} g(s)\,ds$ (+)

$$\Rightarrow \quad y(P) = \frac{1}{2}\left(f(Q) + f(R)\right) + \frac{1}{2c}\int_Q^R g(s)\,ds, \qquad (\dagger\dagger)$$

where Q and R are points on the x-axis as shown.

(43)



- Note deliberate abuse of notation in (H) to aid geometric interpretation of (+).

- Defn: $x \pm ct = x_0 \pm ct_0$ called characteristic lines through P.

- (H) $\Rightarrow$ $y(P)$ depends only on
  (i) f through the values f takes at Q and R;
  (ii) g through the values g takes on x-axis between Q and R.

- Defn: The interval $[x_0 - ct_0, x_0 + ct_0]$ of the x-axis between Q and R is called the domain of dependence of $P:(x_0,t_0)$.

- If f or g modified outside the domain of dependence of P, then $y(P)$ is unchanged.

- Exploit geometric interpretation (H) of DF (+) to construct explicit formulae for the solution: contribution to $y(P)$ from f and g change at points on x-axis where f and g change their analytic behaviour.

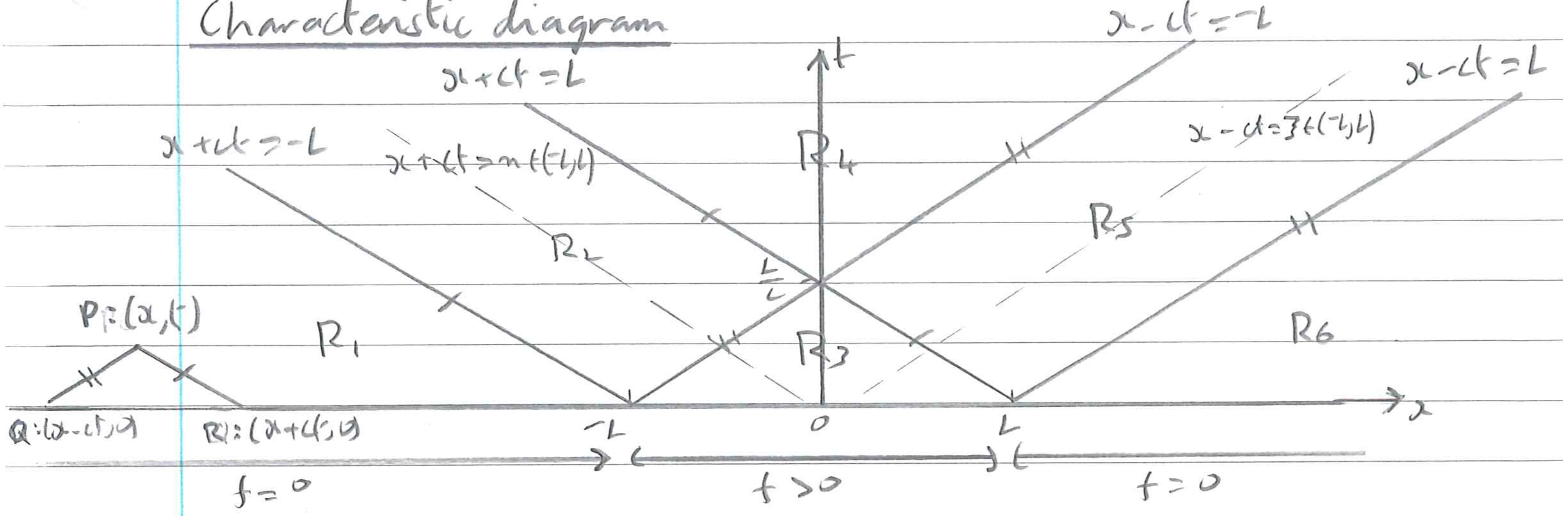
- Hence, given a particular f and g, first task is to identify these points on x-axis and sketch the characteristic lines $x \pm ct = $ constant through each of them – this is the characteristic diagram.

- This divides the (x,t)-plane, with $t>0$, into regions in which the contributions from f and g may be different.

(44)

Back to earlier example...

<u>Characteristic diagram</u>



- DF $\Rightarrow$ $y(P)=\frac{1}{2}\left(f(Q)+f(R)\right)$, where P, Q, R are points shown.

- $PQ \parallel x-ct=\pm L$ and $PR \parallel x+ct=\pm L$, so solution as follows.

- $P\in R_1 \Rightarrow y=\frac{1}{2}[0+0]$

- $P\in R_2 \Rightarrow y=\frac{1}{2}\left[0+\varepsilon\cos^4\left(\frac{\pi}{2L}(x+ct)\right)\right]$

- $P\in R_3 \Rightarrow y=\frac{1}{2}\left[\varepsilon\cos^4\left(\frac{\pi}{2L}(x-ct)\right)+\varepsilon\cos^4\left(\frac{\pi}{2L}(x+ct)\right)\right]$

- $P\in R_4 \Rightarrow y=\frac{1}{2}[0+0]$

- $P\in R_5 \Rightarrow y=\frac{1}{2}\left[\varepsilon\cos^4\left(\frac{\pi}{2L}(x-ct)\right)+0\right]$

- $P\in R_6 \Rightarrow y=\frac{1}{2}[0+0]$

- Since y is continuous on characteristics bounding regions, it does not matter to which region each belongs, e.g. could pick R_1 : $x+ct<-L$, $t>0$;
  R_2 : $-L\le x+ct\le L$, $x-ct\le L$;
  R_3 : $-L<x+ct<L$, $-L<x-ct<L$, $t>0$;

etc.

(45)

<u>Example 4.6</u>: Suppose $y(x,t)$ s.t. ① $y_{tt} = c^2 y_{xx}$ for $-\infty < x < \infty, t > 0$;

② $y(x,0) = f(x), y_t(x,0) = g(x)$ for $-\infty < x < \infty$,

Find $y(x,t)$ when $f(x) = 0$ and $g(x) = \begin{cases} vx/L & \text{for } |x| \leq L, \\ 0 & \text{otherwise,} \end{cases}$

where $L, v \in \mathbb{R}^+$.

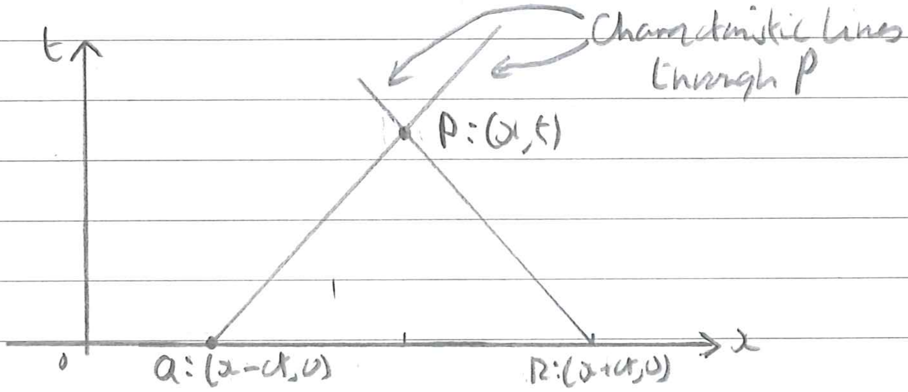
- Recall D'Alembert's Formulae (DF) for the solution of ①-②:

$$y(x,t) = \frac{1}{2}\left(f(x-ct) + f(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

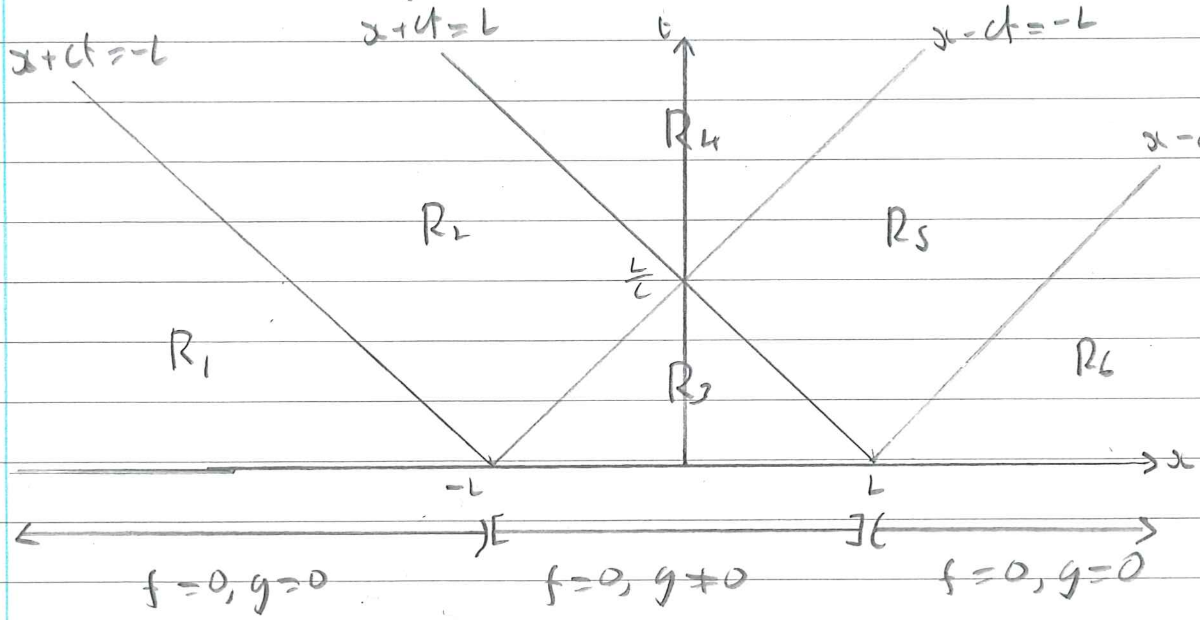
- Thus,

$$y(P) = \frac{1}{2c}\int_{Q}^{R} g(s)\,ds,$$

where P, Q, R are the points shown



<u>Characteristic diagram</u>



(46)

- PQ ∥ $x - ct = \pm L$ and PR ∥ $x + ct = \pm L$, so solution as follows.

$$R_1: \quad y = \frac{1}{2c}\int_{x-ct}^{x+ct} 0 \, ds = 0$$

$$R_2: \quad y = \frac{1}{2c}\int_{x-ct}^{-L} 0 \, ds + \frac{1}{2c}\int_{-L}^{x+ct} \frac{vs}{L} \, ds = \frac{v}{4Lc}\left((x+ct)^2 - L^2\right)$$

$$R_3: \quad y = \frac{1}{2c}\int_{x-ct}^{x+ct} \frac{vs}{L} \, ds = \frac{v}{4Lc}\left((x+ct)^2 - (x-ct)^2\right) = \frac{vxt}{L}$$

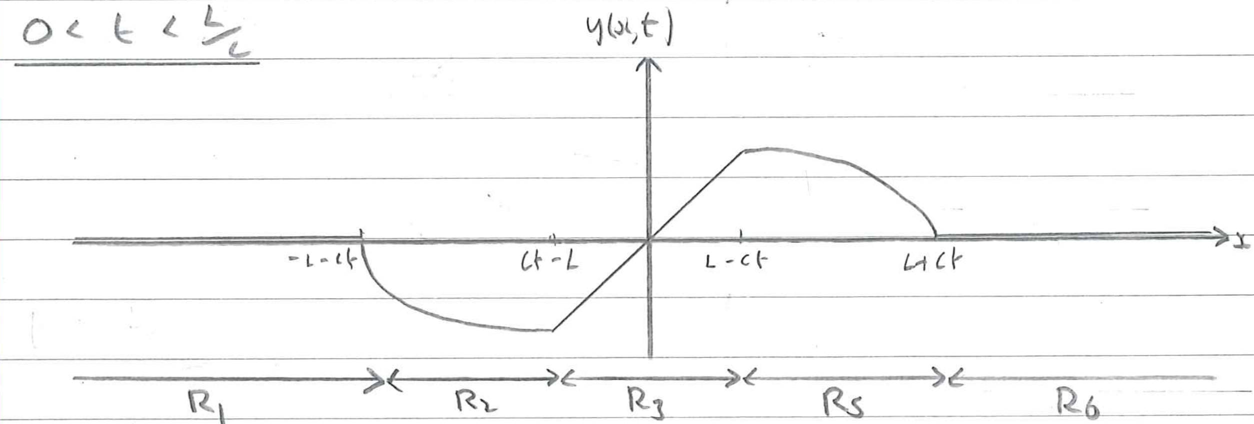
$$R_4: \quad y = \frac{1}{2c}\int_{x-ct}^{-L} 0 \, ds + \frac{1}{2c}\int_{-L}^{L} \frac{vs}{L} \, ds + \frac{1}{2c}\int_{L}^{x+ct} 0 \, ds = 0$$

$$R_5: \quad y = \frac{1}{2c}\int_{x-ct}^{L} \frac{vs}{L} \, ds + \frac{1}{2c}\int_{L}^{x+ct} 0 \, ds = \frac{v}{4Lc}\left(L^2 - (x-ct)^2\right)$$

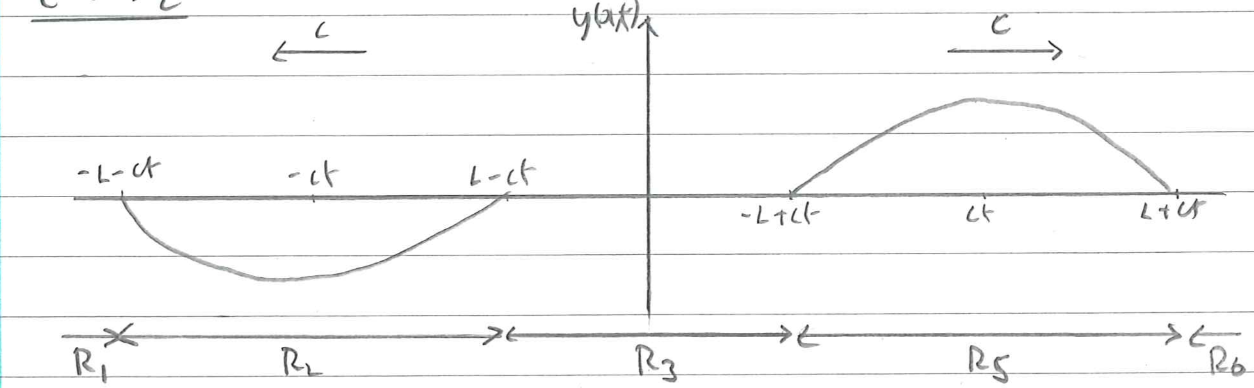
$$R_6: \quad y = \frac{1}{2c}\int_{x-ct}^{x+ct} 0 \, ds = 0$$

- Note solution cts across borders between regions.

<u>$0 < t < L/c$</u>



<u>$t > L/c$</u>



- Note ∃ corners ⇒ not a classical (twice cts diff.) solution!