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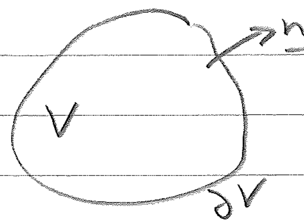
# Laplace's equation in the plane

- Heat conduction in a rigid isotropic material (e.g. metal) is governed in 3D by the heat equation

$$T_t = \kappa \nabla^2 T,$$

where $T(x,y,z,t)$ is the temperature, $\kappa$ the thermal diffusivity and $\nabla^2 T = T_{xx} + T_{yy} + T_{zz}$.

- Derive in Multivariable Calculus from conservation of energy and Fourier's Law using the Divergence Theorem:




$$\frac{d}{dt}\iiint_V \rho c T\, dV = \iint_{\partial V} \underline{q}\cdot(-\underline{n})\, dS \qquad \text{[Energy]}$$

$$\underset{\text{(Div. Thm)}}{\Rightarrow} \iiint_V \rho c T_t\, dV = -\iiint_V \nabla\cdot\underline{q}\, dV$$

$$\Rightarrow \rho c T_t + \nabla\cdot\underline{q} = 0 \text{ (assuming LHS cts)}$$

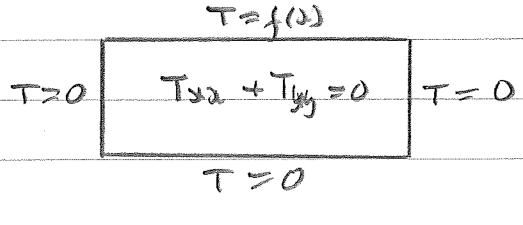
Substitute $\underset{\text{(Fourier's Law)}}{\underline{q} = -k\nabla T} \Rightarrow T_t = \frac{k}{\rho c}\nabla\cdot\nabla T = \kappa\nabla^2 T$ $\square$

- In this course we consider 2D steady-state solutions:

$$T = T(x,y) \Rightarrow \boxed{\begin{array}{c} T_{xx} + T_{yy} = 0 \\ \text{Laplace's equation} \end{array}}$$

## BVP in Cartesian Coordinates

Find $T(x,y)$ s.t. ① $T_{xx} + T_{yy} = 0$ for $0 < x < a$, $0 < y < b$;
② $T(0,y) = 0$, $T(a,y) = 0$ for $0 < y < b$;
③ $T(x,0) = 0$, $T(x,b) = f(x)$ for $0 < x < a$.



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Apply Fourier's method.

Step (I): $T = F(x)G(y)$ in ① $\Rightarrow \frac{F''(x)}{F(x)} = -\frac{G''(y)}{G(y)}$ $\quad (FG \neq 0)$

LHS ind. $y$ & RHS ind. $x \Rightarrow$ LHS = RHS ind. $x$ & $y \Rightarrow$ LHS = RHS = $-\lambda \in \mathbb{R}$, say.

Hence $-F'' = \lambda F$ for $0 < x < a$. ② & $T$ nontrivial $\Rightarrow F(0) = 0$, $F(a) = 0$.

Solved before! Only nontrivial solutions are $F(x) = B\sin\left(\frac{n\pi x}{a}\right)$ ($B \in \mathbb{R}$) for $\lambda = \left(\frac{n\pi}{a}\right)^2$, $n \in \mathbb{N} \setminus \{0\}$.

$\lambda = \frac{n\pi}{a} \Rightarrow G'' - \left(\frac{n\pi}{a}\right)^2 G = 0 \Rightarrow G = C\cosh\left(\frac{n\pi y}{a}\right) + D\sinh\left(\frac{n\pi y}{a}\right)$ $(C, D \in \mathbb{R})$

Combo $\Rightarrow$ nontrivial sep. solns of ①-② are

$$T_n(x,y) = \sin\left(\frac{n\pi x}{a}\right)\left(a_n\cosh\left(\frac{n\pi y}{a}\right) + b_n\sinh\left(\frac{n\pi y}{a}\right)\right),$$

where $a_n, b_n \in \mathbb{R}$ and $n \in \mathbb{N} \setminus \{0\}$.

Step (II): Formally superimpose $\Rightarrow T(x,y) = \sum_{n=1}^{\infty} T_n(x,y)$ is the general series solution of ①-②

Step (III): BC on $y = 0 \Rightarrow 0 = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{a}\right) \Rightarrow a_n = 0 \ \forall n$.

BC on $y = b \Rightarrow f(x) = \sum_{n=1}^{\infty} b_n \sinh\left(\frac{n\pi b}{a}\right)\sin\left(\frac{n\pi x}{a}\right)$ for $0 < x < a$,

so that $b_n \sinh\left(\frac{n\pi b}{a}\right) = \frac{2}{a}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)dx$ by theory of FS.

NB: Could also apply BC on $y = 0$ to find $a_n = 0$ at end of step (I), i.e. before superimposing in step (II).

NB: On sheet consider case in which $a = b = L$ and $f = T^* \in \mathbb{R}$

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## BVP in plane polar coordinates

- In plane polar coordinates $(r,\theta)$, Laplace's equation for $T(r,\theta)$ becomes

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0 \quad \text{for } r>0. \quad (*)$$

- Start by finding all nontrivial separable solutions that are $2\pi$-periodic in $\theta$.

- $T = F(r)G(\theta) \Rightarrow F''G + \frac{1}{r}F'G + \frac{1}{r^2}FG'' = 0$

$$\underset{(\times \frac{r^2}{FG})}{\Rightarrow} \frac{r^2F''(r) + rF'(r)}{F(r)} = -\frac{G''(\theta)}{G(\theta)} \qquad (FG \neq 0)$$

- LHS ind. $\theta$ & RHS ind. $r$ $\Rightarrow$ LHS = RHS ind. $r$ & $\theta$ $\Rightarrow$ LHS = RHS = $\lambda \in \mathbb{R}$.

- Hence, need to find all $\lambda \in \mathbb{R}$ s.t. $G''(\theta) + \lambda G(\theta) = 0$ has a nontrivial solution $G(\theta)$ of period $2\pi$. Consider cases.

(i) $\lambda = -\omega^2$ ($\omega > 0$ wlog) $\Rightarrow G(\theta) = A\cosh(\omega\theta) + B\sinh(\omega\theta)$ $(A, B \in \mathbb{R})$

- $G$ $2\pi$-periodic $\Rightarrow G(0) = G(\pm 2\pi) \Rightarrow A = A\cosh(2\pi\omega) \pm B\sinh(2\pi\omega)$
$\underset{(+,-)}{\Rightarrow} A(\cosh(2\pi\omega) - 1) = 0,\ B\sinh(2\pi\omega) = 0 \underset{(\omega>0)}{\Rightarrow} A = B = 0 \Rightarrow G = 0.$

(ii) $\lambda = 0 \Rightarrow G(\theta) = A + B\theta$ $(A, B \in \mathbb{R})$.

- $G$ $2\pi$-periodic $\Rightarrow B = 0$, but $A$ arbitrary admissible.

- $r^2F'' + rF' = 0 \Rightarrow r(rF')' = 0$
$\Rightarrow rF' = d \qquad (r>0, d \in \mathbb{R})$
$\Rightarrow F = c + d\log r \qquad (c \in \mathbb{R})$

- Combo $\Rightarrow T_0 = A_0 + B_0 \log r \qquad (A_0, B_0 \in \mathbb{R})$
This is a cylindrically symmetric solution (i.e. ind. of $\theta$).

(iii) $\lambda = \omega^2$ ($\omega > 0$ wlog) $\Rightarrow G(\theta) = R\cos(\omega\theta + \Phi)$ $(R, \Phi \in \mathbb{R})$

- $G$ nontrivial $\Rightarrow R \neq 0 \Rightarrow G$ has prime period $\frac{2\pi}{\omega}$.

Hence, $G$ $2\pi$-periodic and non-trivial $\Rightarrow \exists n \in \mathbb{N}\setminus\{0\}$ s.t. $n \cdot \frac{2\pi}{\omega} = 2\pi$, i.e. $\omega = n$ for some $n \in \mathbb{N}\setminus\{0\}$.

In practice, better to write $G(\theta) = A\cos(n\theta) + B\sin(n\theta)$, where $A = R\cos\Phi$, $B = -R\sin\Phi$ are arb. real constants.

- $\lambda = n^2 \Rightarrow r^2F'' + rF' - n^2F = 0$ for $r > 0$. (Euler's ODE)

Let $r = e^t$, $F(r) = W(t)$, then $\frac{dW}{dt} = \frac{dF}{dr}\frac{dr}{dt} = r\frac{dF}{dr}$, so

$$\frac{d^2W}{dt^2} = \frac{d}{dr}\left(r\frac{dF}{dr}\right)\frac{dr}{dt} = r\frac{d}{dr}\left(r\frac{dF}{dr}\right) = r^2F'' + rF' = n^2F = n^2W$$

$W = e^{\mu t} \Rightarrow \mu^2 = n^2 \Rightarrow W$ has general solution $W = Ce^{nt} + De^{-nt}$ $(C, D \in \mathbb{R}) \Rightarrow F$ has general solution

$$F(r) = Cr^n + Dr^{-n} \quad (C, D \in \mathbb{R}).$$

NB: Alternatively, let $F(r) = r^\mu$, then $\mu(\mu-1) + \mu - n^2 = 0$ $\Rightarrow \mu^2 = n^2 \Rightarrow \mu = \pm n \Rightarrow$ general solution as above by theory of linear 2nd order ODEs.

- Combo $\Rightarrow T_n = (A_nr^n + B_nr^{-n})\cos(n\theta) + (C_nr^n + D_nr^{-n})\sin(n\theta)$,

where $A_n = AC$, $B_n = AD$, $C_n = BC$, $D_n = BD$ are arb. real constants and $n \in \mathbb{N}\setminus\{0\}$.

- Superimpose $\Rightarrow$ general series solution of $(*)$ is

$$T(r,\theta) = \sum_{n=0}^{\infty} T_n(r,\theta) = A_0 + B_0\log r + \sum_{n=1}^{\infty}\Big((A_nr^n + B_nr^{-n})\cos(n\theta) + (C_nr^n + D_nr^{-n})\sin(n\theta)\Big).$$

## BVP in plane polar coordinates ctd

- Last lecture we showed that the general series solution of

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0 \qquad (r>0)$$

is given by

$$T(r,\theta) = A_0 + B_0 \log r + \sum_{n=1}^{\infty}\left((A_n r^n + B_n r^{-n})\cos(n\theta) + (C_n r^n + D_n r^{-n})\sin(n\theta)\right) \qquad (*)$$

where $A_n, B_n, C_n, D_n \in \mathbb{R}$.

<u>Example 5.1</u>: Find $T$ s.t. ① $\nabla^2 T = 0$ in $a<r<b$,
② $T = T_0^*$ on $r=a$, $T = T_1^*$ on $r=b$

where $a, b, T_0^*, T_1^* \in \mathbb{R}$.

- ① $\Rightarrow$ $(*)$ pertains. BCs ② can only be satisfied if

$$T_0^* = A_0 + B_0 \log a + \sum_{n=1}^{\infty}\left((A_n a^n + B_n a^{-n})\cos(n\theta) + (C_n a^n + D_n a^{-n})\sin(n\theta)\right)$$

$$T_1^* = A_0 + B_0 \log b + \sum_{n=1}^{\infty}\left((A_n b^n + B_n b^{-n})\cos(n\theta) + (C_n b^n + D_n b^{-n})\sin(n\theta)\right)$$

each for $-\pi < \theta \le \pi$, say.

- Since the Fourier coefficients of a Fourier series are unique, we can equate them on LHS & RHS of each equality $\Rightarrow$

$$\begin{bmatrix} 1 & \log a \\ 1 & \log b \end{bmatrix}\begin{bmatrix} A_0 \\ B_0 \end{bmatrix} = \begin{bmatrix} T_0^* \\ T_1^* \end{bmatrix}, \quad \underbrace{\begin{bmatrix} a^n & a^{-n} \\ b^n & b^{-n} \end{bmatrix}\begin{bmatrix} A_n \\ B_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} a^n & a^{-n} \\ b^n & b^{-n} \end{bmatrix}\begin{bmatrix} C_n \\ D_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{n \in \mathbb{N}\setminus\{0\}}$$

$$\underset{(a<b)}{\Rightarrow} \quad \begin{bmatrix} A_0 \\ B_0 \end{bmatrix} = \frac{1}{\log(b/a)}\begin{bmatrix} \log b & -\log a \\ -1 & 1 \end{bmatrix}\begin{bmatrix} T_0^* \\ T_1^* \end{bmatrix}, \quad \underbrace{\begin{bmatrix} A_n \\ B_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} C_n \\ D_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},}_{n \in \mathbb{N}\setminus\{0\}}$$

$$\Rightarrow T = \frac{T_0^* \log b - T_1^* \log a}{\log(b/a)} + \frac{T_1^* - T_0^*}{\log(b/a)} \log r$$

Dimensionally correct!?

$$T = T_0^* \frac{\log\left(\frac{r}{b}\right)}{\log\left(\frac{a}{b}\right)} + T_1^* \frac{\log\left(\frac{r}{a}\right)}{\log\left(\frac{b}{a}\right)} \quad \checkmark$$

$\square$

NB: Alternatively, we could have sought a circularly symmetric solution $T = T(r)$ from the outset because boundary data is ind. of $\theta$. However, method above generalizes to $T_0^*$ and $T_1^*$ being functions of $\theta$.

<u>Example 5.2</u>: Find $T$ s.t. ① $\nabla^2 T = 0$ in $r < a$,
② $T(a, \theta) = T^* \sin^3\theta$ for $-\pi < \theta \le \pi$,
where $a, T^* \in \mathbb{R}^+$.

- ① $\Rightarrow$ $T$ must be twice differentiable wrt $x$ and $y$ at origin

  $\Rightarrow$ $T$ must certainly be cts and therefore bounded at origin

  $\Rightarrow$ (*) pertains but with $B_n = 0 \ \forall n \in \mathbb{N}$ and $D_n = 0 \ \forall n \in \mathbb{N}\cup\{0\}$

- ② then requires:

$$T^* \sin^3\theta = A_0 + \sum_{n=1}^{\infty} \left(A_n a^n \cos(n\theta) + B_n a^n \sin(n\theta)\right) \text{ for } -\pi < \theta \le \pi$$

But the FS for the LHS is given by the identity

$$T^* \sin^3\theta = \frac{3T^*}{4} \sin\theta - \frac{T^*}{4} \sin(3\theta),$$

so equating Fourier coefficients gives

$$B_1 a = \frac{3T^*}{4}, \quad B_3 a^3 = -\frac{T^*}{4} \quad \text{and rest vanish.}$$

Hence, $T = \frac{3T^*}{4}\left(\frac{r}{a}\right)\sin\theta - \frac{T^*}{4}\left(\frac{r}{a}\right)^3 \sin 3\theta$

$Q^{\underline{n}}$: What is the heat flux out of the disc through $r=a$?

$A^{\underline{ns}}$: Outward pointing unit normal $\underline{n} = \underline{e}_r$, so by Fourier's Law

$$\underline{q}\cdot\underline{n}\big|_{r=a} = (-k\nabla T)\cdot\underline{e}_r\big|_{r=a} = -kT_r(a,\theta) = -k\left(\frac{3T^*}{4a}\sin\theta - \frac{3T^*}{4a}\sin 3\theta\right)$$

NB: $\nabla^2 T = 0 \Leftrightarrow \nabla\cdot\underline{q} = 0 \Rightarrow \int_{r=a} \underline{q}\cdot\underline{n}\, ds = \iint_{r<a} \nabla\cdot\underline{q}\, dx dy = 0$,

so zero net heat flux through $r=a$ as there's no volumetric heating

## Poisson's Integral Formula

- Find $T$ s.t. $\nabla^2 T = 0$ in $r<a$ with $T(a,\theta) = f(\theta)$ for $-\pi<\theta\le\pi$, where $a\in\mathbb{R}^+$ and $f$ is given.

- As in last example, general series solution is given by $(*)$ with $B_n = 0\ \forall n\in\mathbb{N}$, $D_n = 0\ \forall n\in\mathbb{N}\setminus\{0\}$, so BC satisfied if

$$f(\theta) = \underbrace{A_0}_{a_0/2} + \sum_{n=1}^{\infty}\left(\underbrace{A_n a^n}_{a_n}\cos(n\theta) + \underbrace{B_n a^n}_{b_n}\sin(n\theta)\right) \text{ for } -\pi<\theta\le\pi.$$

- Theory of FS then gives

$$A_0 = \frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)\,d\phi$$

$$A_n = \frac{a_n}{a^n} = \frac{1}{\pi a^n}\int_{-\pi}^{\pi} f(\phi)\cos(n\phi)\,d\phi \qquad (n\in\mathbb{N}\setminus\{0\})$$

$$B_n = \frac{b_n}{a^n} = \frac{1}{\pi a^n}\int_{-\pi}^{\pi} f(\phi)\sin(n\phi)\,d\phi \qquad (n\in\mathbb{N}\setminus\{0\})$$

where we introduced a dummy variable $\phi$ for convenience below.

- Given a particular $f$, can evaluate these expressions (see e.g. Example 5.2 in online notes), but remarkably we can sum for general $f$ (suff. smooth that following OK).

- Sub. Fourier coeffs into general series soln and assume $\sum \int = \int \sum$ gives

$$T(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)\,d\phi + \sum_{n=1}^{\infty}\left(\frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{r}{a}\right)^n\left(\cos(n\theta)\cos(n\phi)+\sin(n\theta)\sin(n\phi)\right)f(\phi)\,d\phi\right)$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos n(\theta-\phi)\right) f(\phi)\,d\phi$$

- Let $\alpha = \theta - \phi$ and $z = \frac{r}{a}e^{i\alpha}$, then

$$\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos(n\alpha) = \mathrm{Re}\left(\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n e^{in\alpha}\right)$$

$$= \mathrm{Re}\left(\frac{1}{2} + \sum_{n=1}^{\infty} z^n\right)$$

$$= \mathrm{Re}\left(\frac{1}{2} + \frac{z}{1-z}\right) \qquad (|z|<1)$$

$$= \frac{1}{2}\mathrm{Re}\left(\frac{1+z}{1-z}\right)$$

$$= \frac{a^2 - r^2}{2(a^2 - 2ar\cos\alpha + r^2)} \qquad \left(z = \frac{a}{r}e^{i\alpha}\right)$$

- Hence, obtain PIF:

$$T(r,\theta) = \frac{(a^2-r^2)}{2\pi}\int_{-\pi}^{\pi}\frac{f(\phi)\,d\phi}{a^2 - 2ar\cos(\theta-\phi)+r^2} \qquad (r<1)$$

- NB: $r = 0 \Rightarrow T = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)\,d\phi$

This means temp at centre of disc is average of the temp. profile on the boundary.

- Next time: uniqueness.