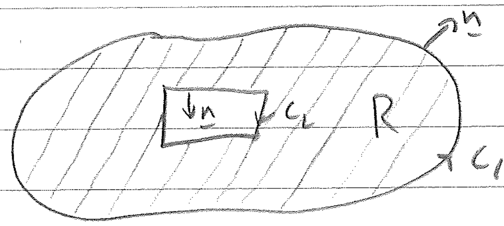
Uniqueness

(Handout) **Green's Theorem in the plane (Divergence Theorem Form)**

Let R be a closed bounded region in the (x,y)-plane, whose boundary ∂R is the union $C_1 \cup C_2 \cup \ldots \cup C_m$ of a finite number of piecewise smooth simple closed curves.



Let $\underline{F} = (F_1(x,y), F_2(x,y))$ be continuous and have continuous first order derivatives on $R \cup \partial R$. Then

$$\iint_R \nabla \cdot \underline{F}\, dx\,dy = \int_{\partial R} \underline{F} \cdot \underline{n}\, ds,$$

where $\underline{n}$ is the outward pointing unit normal to ∂R in the (x,y)-plane and ds an element of arclength.

Example: Derivation of the 2D inhomogeneous heat equation

$$[\text{Energy}]: \quad \underbrace{\frac{d}{dt}\iint_R \rho c T\, dx\,dy}_{\substack{\text{Rate of change of}\\ \text{internal heat energy}}} = \underbrace{\int_{\partial R} \underline{q} \cdot (-\underline{n})\, ds}_{\substack{\text{Net heat flux}\\ \text{into } R \text{ through } \partial R}} + \underbrace{\iint_R Q\, dx\,dy}_{\substack{\text{Volumetric}\\ \text{heating}}}$$

NB: [Each term] $= \mathrm{J\,m^{-1}\,s^{-1}}$ since this is per unit distance in z-direction.

Assuming T_t continuous on $R \cup \partial R$ & using Green's Thm with $\underline{F} = \underline{q}$ gives
$$\iint_R \rho c T_t + \nabla \cdot \underline{q} - Q\, dx\,dy = 0$$

Assuming integrand continuous, R arbitrary $\Rightarrow \rho c T_t + \nabla \cdot \underline{q} = Q$

Finally, Fourier's law $\underline{q} = -k\nabla T$ gives $\rho c T_t = \nabla \cdot (k \nabla T) + Q$ $\square$

Uniqueness for the Dirichlet problem

Thm: Suppose $T(x,y)$ s.t. $\nabla^2 T = 0$ in R with $T = f$ on ∂R (Dirichlet problem), where R as in Green's Thm and path-connected and f given. Then the BVP has at most one solution.

Pf: Let W be the difference between two solutions, then linearity gives ① $\nabla^2 W = 0$ in R;
② $W = 0$ on ∂R.

Trick: let $\underline{F} = W \nabla W = \nabla(\frac{1}{2}W^2)$ in Green's Thm.

Then $$\iint_R \nabla \cdot (W\nabla W)\, dx\,dy = \int_{\partial R} W \nabla W \cdot \underline{n}\, ds$$

But ① $\Rightarrow$ $\nabla \cdot (W\nabla W) = W\nabla^2 W + \nabla W \cdot \nabla W = |\nabla W|^2$ in R

② $\Rightarrow$ $W \nabla W \cdot \underline{n} = 0$ on ∂R

so $$\iint_R |\nabla W|^2\, dx\,dy = 0.$$

Assuming ∇W is continuous on $R \cup \partial R$, this implies $\nabla W = \underline{0}$ on $R \Rightarrow W =$ constant on R (as it's path-connected).

But $W = 0$ on ∂R, so assuming W is continuous on $R \cup \partial R$, the constant must vanish, so that $W = 0$ on $R \cup \partial R$ $\square$

Example: Find T s.t. $\nabla^2 T = 0$ in $r < a$ with $T = T^* \frac{x}{a}$ on $r = a$

If we can find any solution, then uniqueness thm guarantees it is the only solution.

Could proceed via general series solution or Poisson's Integral Formula, but quicker to spot $T = T^* \frac{x}{a}$.

<u>Example</u>: Find T s.t. $\nabla^2 T = 0$ in $r > a$ with $T = T^* \frac{x}{a}$ on $r = a$ and T bounded as $r \to \infty$.

Spot $T = B_1 r^{-1} \cos\theta$ is a solution provided $B_1 a^{-1} = T^*$.

<u>Qn</u>: Is it the only solution?

<u>Ans</u>: Uniqueness thm above not applicable because R not bdd.

But if W is difference between two solutions, then for fixed $b > a$

$$\iint_{a<r<b} |\nabla W|^2 \,dx\,dy \underset{\substack{\nabla^2 W = 0 \text{ in} \\ a<r<b}}{=} \iint_{a<r<b} \nabla\cdot(W\nabla W)\,dx\,dy = \int_{r=b} \underbrace{W\nabla W\cdot \underline{e}_r\,ds}_{W\frac{\partial W}{\partial r} r\,d\theta} - \underbrace{\int_{r=a} W\nabla W\cdot\underline{e}_r\,ds}_{=0 \text{ since } W=0 \text{ at } r=a}$$

So have uniqueness provided $rW\frac{\partial W}{\partial r} \to 0$ as $r \to \infty$, which is the case if e.g. $r\frac{\partial T}{\partial r} \to 0$ as $r \to \infty$. $\square$

<u>Uniqueness for the Neumann Problem</u>

<u>Thm</u>: Suppose $T(x,y)$ s.t. $\nabla^2 T = F$ in R with $\frac{\partial T}{\partial n} \equiv \underline{n}\cdot\nabla T = g$ on ∂R (Neumann problem), where R as in Green's Thm and path-connected and F, g given. Then the BVP has no solution unless

$$\iint_R F\,dx\,dy = \int_{\partial R} g\,ds.$$

When a solution exists, it is not unique: any two solutions differ by a constant.

<u>Pf</u>: Suppose there is a solution T and let $\underline{F} = \nabla T$ in Green's Thm, then

$$\iint_R F\,dx\,dy = \iint_R \nabla\cdot(\nabla T)\,dx\,dy = \int_{\partial R} \nabla T\cdot\underline{n}\,ds = \int_{\partial R} g\,ds.$$

Now let W be the difference between two solutions, so that $\nabla^2 W = 0$ in R and $\frac{\partial W}{\partial n} = 0$ in R. Then

$$\iint_R |\nabla W|^2 \, dx\,dy \underset{\text{PDE}}{=} \iint_R \nabla\cdot(W\nabla W)\,dx\,dy \underset{\text{Green's Thm}}{=} \int_{\partial R} W\nabla W.\underline{n}\,ds \underset{\text{BC}}{=} 0 \text{ as before.}$$

Assuming ∇W is cts on $R \cup \partial R$, this implies $\nabla W = \underline{0}$ on R, so that $W =$ constant on R. Hence, $W =$ constant on $R \cup \partial R$ assuming W cts there. □

__Example__: Find T s.t. $\nabla^2 T = 0$ in $r < a$ with $\frac{\partial T}{\partial n} = g(\theta)$ on $r = a$, where g is given.

General series solution of $\nabla^2 T = 0$ in $r < a$ is

$$T = A_0 + \sum_{n=1}^{\infty} \left(A_n r^n \cos(n\theta) + B_n r^n \sin(n\theta)\right).$$

On $r = a$, $\frac{\partial T}{\partial n} = \underline{n}.\nabla T = \frac{\partial T}{\partial r}$, so BC can satisfied only if

$$g(\theta) = \sum_{n=1}^{\infty} \left(nA_n a^{n-1}\cos(n\theta) + nB_n a^{n-1}\sin(n\theta)\right) \text{ for } -\pi < \theta \leq \pi \text{ (say)}$$

Theory of FS then requires

$$0 = \frac{1}{\pi}\int_{-\pi}^{\pi} g(\theta)\,d\theta \qquad (+)$$

$$nA_n a^{n-1} = \frac{1}{\pi}\int_{-\pi}^{\pi} g(\theta)\cos(n\theta)\,d\theta \qquad (n \in \mathbb{N}\setminus\{0\})$$

$$nB_n a^{n-1} = \frac{1}{\pi}\int_{-\pi}^{\pi} g(\theta)\sin(n\theta)\,d\theta \qquad (n \in \mathbb{N}\setminus\{0\})$$

Two cases.

Either (i) g s.t. $(+)$ not true, in which case there is no solution;

or (ii) g s.t. $(+)$ is true, in which case the solution is not unique (since A_0 arb., while rest of Fourier coeffs are uniquely determined).

This agrees with uniqueness thm, which guarantees that in case (ii) we've found all possible solutions

Wellposedness

- An IBVP or BVP is wellposed if $\exists!$ solution that depends continuously on the data in ICs and/or BCs.

Example: Wave Equation

- Suppose $y(x,t)$ s.t. ① $y_{tt} = y_{xx}$ for $-\infty < x < \infty$, $t > 0$,
② $y(x,0) = f(x)$, $y_t(x,0) = g(x)$ for $-\infty < x < \infty$,
where f and g are given.

- By D'Alembert's Formula $\exists!$ solution since ① – ② $\Rightarrow$

$$y(x,t) = \tfrac{1}{2}\left(f(x-t) + f(x+t)\right) + \tfrac{1}{2}\int_{x-t}^{x+t} g(s)\,ds.$$

- Now change data f, g to F, G and let Y be new solution:

$$Y(x,t) = \tfrac{1}{2}\left(F(x-t) + F(x+t)\right) + \tfrac{1}{2}\int_{x-t}^{x+t} G(s)\,ds.$$

- Suppose $\exists\, \delta > 0$ s.t. $|f(x) - F(x)| < \delta$, $|g(x) - G(x)| < \delta$ $\forall x \in \mathbb{R}$. $(+)$

Then,
$$|y(x,t) - Y(x,t)| = \left|\tfrac{1}{2}\left(f(x-t) - F(x-t)\right) + \tfrac{1}{2}\left(f(x+t) - F(x+t)\right) + \tfrac{1}{2}\int_{x-t}^{x+t} g(s) - G(s)\,ds\right|$$
$$\leq \tfrac{1}{2}\left|f(x-t) - F(x-t)\right| + \tfrac{1}{2}\left|f(x+t) - F(x+t)\right| + \tfrac{1}{2}\int_{x-t}^{x+t} |g(s) - G(s)|\,ds$$
$$\leq \tfrac{1}{2}\delta + \tfrac{1}{2}\delta + \tfrac{1}{2}\cdot 2t \cdot \delta$$
$$= (1+t)\delta \quad \text{for } -\infty < x < \infty,\ t \geq 0 \quad (++)$$

- Fix any $T > 0$ and any $\varepsilon > 0$.

If we pick $\delta = \frac{\varepsilon}{1+T}$ in $(+)$, then $(++)$ implies

$$|y(x,t) - Y(x,t)| \leq \varepsilon \frac{1+t}{1+T} < \varepsilon \quad \text{for } -\infty < x < \infty,\ 0 < t < T$$

(63)

In this sense the solution depends continuously on the data and the IVP is well-posed.

Example : Try IVP for Laplace's equation!

- Suppose $y(x,t)$ s.t. ① $y_{xx} + y_{tt} = 0$ for $-\infty < x < \infty$, $t > 0$,
  ② $y(x,0) = f(x)$, $y_t(x,0) = g(x)$ for $-\infty < x < \infty$,

where f and g are given.

- Problem (I) : $f_1 = 0$, $g_1 = 0 \Rightarrow y_1 = 0$ is a solution.

  Problem (II) : $f_2 = 0$, $g_2 = \delta \cos(\frac{x}{\delta}) \Rightarrow y_2 = \delta^2 \cos(\frac{x}{\delta}) \sinh(\frac{t}{\delta})$ is a solution for any $\delta > 0$.

- Observe that $|f_1(x) - f_2(x)| = 0$, $|g_1(x) - g_2(x)| \leq \delta \; \forall x \in \mathbb{R}$.

  But $|y_1(0,t) - y_2(0,t)| = \delta^2 \sinh(\frac{t}{\delta}) \to \infty$ as $\delta \to 0+$ for any fixed $t > 0$, so cannot make $|y_1(0,t) - y_2(0,t)| < \varepsilon$ for all $0 < t < T$ by taking δ suitably small.

- IVP for Laplace's equation is not well-posed – called ill-posed!

Summary

1. Introduced theory of Fourier Series.
   - Periodic, even & odd functions & periodic extensions;
   - Euler's formulae for Fourier coeffs via orthogonality relations;
   - Statement of a powerful pointwise convergence thm;
   - Related rate of convergence to smoothness;
   - Discussed Gibb's phenomenon – try to avoid!

2. Heat equation
   - Derivation in 1D, 2D & 3D;
   - Simple solutions (fundamental & root cellar);

64

- units & nondimensionalization;
- Fourier's method for IBVPs;
- Generalized to inhomogeneous heat equation & BCs.
- Uniqueness.

3. Wave equation
- Derivation in 1D with gravity & air resistance;
- normal modes and natural frequencies;
- Fourier's method for IBVPs – plucked & flicked strings;
- Forced & damped wave equation with inhomogeneous BCs;
- normal modes for composite and weighted string;
- D'Alembert's solution and characteristic diagrams.
- Uniqueness.

4. Laplace's equation
- Derivation in 2D & 3D;
- Fourier's method for BVPs in (x, y) & (r, θ);
- Poisson's Integral Formula for Dirichlet problem on a disc;
- Uniqueness of Dirichlet problem;
- Nonexistence & nonuniqueness of Neumann problem

5. Well-posedness
- Introduced concepts developed later on in course.

- Problem sheet questions not too far from prelims questions, so should set you up well for exam.

- Should try at least 3-5 past papers, but maybe avoid the TT 2015 paper – it turned out much tougher than anticipated.

- Collect feedback.