MULTIVARIABLE CALCULUS HT19 SHEET 1
Multiple planar integrals. Change of variables.

1. Let $a, b > 0$. By swapping the order of double integrals of e^{-xy} on the set $[0,\infty) \times [a,b]$ show that

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\,\mathrm{d}x = \ln\left(\frac{b}{a}\right).$$

2. Show that

$$\int_{y=0}^{y=1}\left(\int_{x=0}^{x=1} \frac{x-y}{(x+y)^3}\,\mathrm{d}x\right)\mathrm{d}y = -\frac{1}{2} \neq \frac{1}{2} = \int_{x=0}^{x=1}\left(\int_{y=0}^{y=1} \frac{x-y}{(x+y)^3}\,\mathrm{d}y\right)\mathrm{d}x.$$

3. Show that

$$\iint_{[0,1]^2} |x-y|\,\mathrm{d}A = \frac{1}{3}.$$

Why does this integral represent the mean distance between two points chosen randomly and independently from the unit interval $[0,1]$?

4. (i) The *moment of inertia* of a uniform disc D of radius a and mass m about an axis vertically through its centre equals

$$I_0 = \iint_D r^2\,\rho\,\mathrm{d}A,$$

where ρ is the density of the disc. Determine ρ in terms of a and m. Use polar co-ordinates to show that I_0 equals $ma^2/2$.

(ii) The vertical axis is now moved from the centre of the disc to a point distance $R \leqslant a$ away. The moment of inertia now equals

$$I_R = \iint_D d(r,\theta,R)^2\,\rho\,\mathrm{d}A$$

where $d(r,\theta,R)$ is the distance of the point (r,θ) from the axis. Find I_R and show that it is an increasing function of R for $0 \leqslant R \leqslant a$.

5. Let

$$I = \iint_{[0,1]^2} \frac{\mathrm{d}x\,\mathrm{d}y}{1-xy}.$$

(i) By applying the binomial theorem to the integrand, show that

$$I = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

(ii) Set $x = u - v$ and $y = u + v$. Determine the Jacobian $\partial(x,y)/\partial(u,v)$. Show that

$$I = 4\iint_T \frac{\mathrm{d}u\,\mathrm{d}v}{1-u^2+v^2}$$

where T is the triangle with vertices $(0,0)$, $(1,0)$ and $(1/2,1/2)$.

(iii) By splitting the interval $0 \leqslant u \leqslant 1$ into halves, and using trigonometric substituions and identities, show that $I = \pi^2/6$.

6. (Optional) (i) Let $a \geqslant 0$. By applying the method of Exercise 1 to $e^{-yx} \sin x$, show that

$$\int_0^\infty \left(1 - e^{-ax}\right) \frac{\sin x}{x} \, \mathrm{d}x = \tan^{-1} a.$$

Deduce that $\int_0^\infty \frac{\sin x}{x} \, \mathrm{d}x = \pi/2$.

(ii) What is $\int_0^\infty \frac{\sin cx}{x} \, \mathrm{d}x$ where c is a real number?

(iii) Show that the integrals $\int_0^{n\pi} \left|\frac{\sin x}{x}\right| \mathrm{d}x$ increase without bound as n becomes large.