MULTIVARIABLE CALCULUS HT19 SHEET 2
Multiple integrals. Change of variables.

1. Express the volume of a cone, with height h and base radius a, as a triple integral, and hence show that the cone's volume equals $\pi a^2 h/3$.

2. Show that

$$\iiint_T e^{-x-y-z}\,\mathrm{d}V = 1 - \frac{5}{e^2}.$$

where T is the tetrahedron with vertices $\mathbf{0}, 2\mathbf{i}, 2\mathbf{j}$ and $2\mathbf{k}$.

3. The *mean value* of a function f defined on a region R is given by the formula

$$\mu = \frac{1}{\mathrm{Vol}(R)} \iiint\limits_R f\,\mathrm{d}V.$$

Find the mean value of the function $x^2 + y^2 + z^2$ in the cylindrical region R given by

$$R = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leqslant 1,\ -1 \leqslant z \leqslant 1\}.$$

Find the *median* of the function f; this is defined to be the value h such that

$$\mathrm{Vol}\left(\{(x, y, z) \in R : f(x, y, z) \leqslant h\}\right) = \frac{1}{2}\mathrm{Vol}(R).$$

4. Matter occupies the sphere $x^2 + y^2 + z^2 \leqslant a^2$, with the upper hemisphere being k times as dense as the lower hemisphere. Find the centre of mass of the sphere.

Does your answer make sense when $k = 1$? Where is the centre of mass of a uniform hemisphere?

5. Find the volume of the region which lies in the octant $x > 0$, $y > 0$, $z > 0$ and for which

$$a \leqslant \sqrt{yz} \leqslant b, \qquad a \leqslant \sqrt{zx} \leqslant b, \qquad a \leqslant \sqrt{xy} \leqslant b, \qquad \text{where } 0 < a < b.$$

6. (Optional) For real constants a, b, c, show that

$$\iiint\limits_R x e^{ax+by+cz}\,\mathrm{d}V = \frac{4\pi a}{|\mathbf{a}|^5}\left(\left(3 + |\mathbf{a}|^2\right)\sinh|\mathbf{a}| - 3|\mathbf{a}|\cosh|\mathbf{a}|\right),$$

where R is the region $x^2 + y^2 + z^2 \leqslant 1$ and $|\mathbf{a}| = \sqrt{a^2 + b^2 + c^2}$.