

MULTIVARIABLE CALCULUS HT19 SHEET 3
Surface integrals. Flux Integrals. Solid Angle.

1. Parametrize the surface of a cone, with base radius a and height h , and so express its surface area as a multiple integral. Hence show its surface area equals

$$\pi a \sqrt{a^2 + h^2}.$$

Re-evaluate this surface area by considering how a sector of a disc may be folded to make a cone.

2. Show that the solid angle at the apex of a cone with semiangle α is $2\pi(1 - \cos \alpha)$.

If a sphere has radius R and its centre at distance D from an observer, with $D \gg R$, show that the sphere occupies, as a fraction

$$\frac{1}{2} \left(1 - \frac{\sqrt{D^2 - R^2}}{D} \right) \approx \frac{R^2}{4D^2}$$

of the observer's view.

Use this to explain how the sun (at radius 7×10^5 km and distance 1.5×10^8 km) and moon (at radius 1.8×10^3 km and distance 3.8×10^5 km) occupy roughly the same amount of the sky.

3. Evaluate

$$\iint_{\Sigma} \mathbf{F} \cdot d\mathbf{S}$$

where $\mathbf{F} = ((x-1)x^2y, (y-1)^2xy, z^2-1)$ and Σ is the surface of the unit cube $[0, 1]^3$.

4. Two points are chosen at random on the surface of the sphere Σ with equation $x^2 + y^2 + z^2 = a^2$. Explain why the mean distance between the points equals the integral

$$\mu = \frac{1}{4\pi a^2} \iint_{\Sigma} \sqrt{x^2 + y^2 + (z-a)^2} dS$$

and hence determine μ .

5. (i) Calculate the surface integrals $\iint_{\Sigma} f dS$ and $\iint_{\Sigma} f d\mathbf{S}$ where

$$f(x, y, z) = (x^2 + y^2 + z^2)^2$$

and

$$\Sigma = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = z^2, y \geq 0, 0 \leq z \leq 2\}.$$

(ii) Parametrise the various parts of the boundary $\partial\Sigma$ and determine $\int_{\partial\Sigma} f ds$ and $\int_{\partial\Sigma} f d\mathbf{r}$.

6. (Optional) A spherical shell Σ with equation $x^2 + y^2 + z^2 = 1$ has density $\rho(x, y, z) \geq 0$. Show that its moment of inertia about an axis through the points $(\pm a, \pm b, \pm c)$ on the shell equals

$$I(a, b, c) = \iint_{\Sigma} (1 - (ax + by + cz)^2) \rho(x, y, z) dS.$$

Find this value when ρ is constant. Conversely, if $I(a, b, c)$ is constant for all $(a, b, c) \in \Sigma$, need $\rho(x, y, z)$ be constant?