MULTIVARIABLE CALCULUS HT19 SHEET 5
Green's theorems. Divergence theorem.

1. (i) Let $\mathbf{F}(x,y,z) = (3x^2y^2z, 2x^3yz, x^3y^2)$. Show that $\nabla \wedge \mathbf{F} = \mathbf{0}$ and find a potential ϕ such that $\mathbf{F} = \nabla\phi$. To what extent is ϕ unique?

Verify by direct calculation that

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \phi(\mathbf{q}) - \phi(\mathbf{p})$$

where $\mathbf{p} = (0,0,0)$, $\mathbf{q} = (1,1,1)$ and C is the *twisted cubic* $\mathbf{r}(t) = (t, t^2, t^3)$ with $0 \leqslant t \leqslant 1$.

(ii) Let $\mathbf{F}(x,y,z) = (0, xy-1, y-xz)$. Show that $\nabla \cdot \mathbf{F} = 0$ and that $\mathbf{f}(x,y,z) = (xyz, xy, x)$ is a vector potential – that is $\mathbf{F} = \nabla \wedge \mathbf{f}$. To what extent is $\mathbf{f}$ unique?

2. Use Green's theorem to find the simple closed curve C in the xy-plane that maximises the integral

$$\int_C y^3\,\mathrm{d}x + \left(3x - x^3\right)\,\mathrm{d}y,$$

and determine this maximum.

3. Verify the divergence theorem where $\mathbf{F}(x,y,z) = (y, xy, -z)$, and R is the region enclosed below the plane $z = 4$,and the paraboloid $z = x^2 + y^2$.

4. Let f be a smooth scalar field defined on a region $R \subseteq \mathbb{R}^3$ with a smooth boundary ∂R. Show that

$$\iint_{\partial R} f\mathbf{r} \wedge \mathrm{d}\mathbf{S} = \iiint_{R} \mathbf{r} \wedge \nabla f\,\mathrm{d}V.$$

5. Let R be the region $1 < a < r < b$, where r is the distance from the origin in $\mathbb{R}^2$. Find a solution of the boundary-value problem

$$\nabla^2 f + 1 = 0 \text{ in } R, \qquad \frac{\partial f}{\partial n} + f = 0 \text{ on } \partial R,$$

which is a function of r only. Show that this is the only solution, even within the class of not necessarily radial functions.

6. (Optional) For $-1 < \rho < 1$, let

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{\frac{2\rho xy - x^2 - y^2}{2(1-\rho^2)}\right\}.$$

(i) By rotating the xy-axes appropriately, show that $f_{X,Y}(x,y)$ is a probability density function on $\mathbb{R}^2$. [You may assume the result from lectures that $\int_{-\infty}^{\infty} e^{-at^2}\,\mathrm{d}t = \sqrt{\pi/a}$ where $a > 0$.]

(ii) The marginal distribution X has pdf $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}y$. Show that f_X is the pdf of the normal distribution with mean 0 and standard deviation 1.

(iii) Show that

$$\rho = \mathrm{Cov}(X,Y) = E[XY].$$