MULTIVARIABLE CALCULUS HT19 SHEET 6
Divergence theorem. Examples. Consequences.

1. Let C be a closed, positively oriented curve in $\mathbb{R}^2$ bounding a region D. Show that

$$\text{area of } D = \frac{1}{2}\int_C x\,\mathrm{d}y - y\,\mathrm{d}x.$$

Hence find the area of the ellipse $x^2/a^2 + y^2/b^2 = 1$.

2. Let R be a closed bounded region, bounded by a closed surface ∂R. Let ϕ, ψ be smooth scalar fields on R. Use the divergence theorem to prove:

(i) **Green's First Theorem:**

$$\iiint_R (\psi\nabla^2\phi + \nabla\phi\cdot\nabla\psi)\,\mathrm{d}R = \iint_{\partial R} \psi\nabla\phi\cdot \mathrm{d}\mathbf{S}.$$

(ii) **Green's Second Theorem:**

$$\iiint_R (\psi\nabla^2\phi - \phi\nabla^2\psi)\,\mathrm{d}R = \iint_{\partial R} (\psi\nabla\phi - \phi\nabla\psi)\cdot \mathrm{d}\mathbf{S}.$$

3. Let $D \subseteq \mathbb{R}^2$ be a closed, boundary region with smooth boundary ∂D, and f be a smooth function defined in D. By applying Green's theorem in the plane with suitable functions P and Q, show that

$$\iint_D \nabla^2 f\,\mathrm{d}x\,\mathrm{d}y = \int_{\partial D} \frac{\partial f}{\partial n}\,\mathrm{d}s.$$

4. The temperature $T(r,\theta)$ in an annulus $a \leqslant r \leqslant b$ satisfies $\nabla^2 T = 1$ inside the annulus. On the inner boundary $\partial T/\partial n = k$, where $k > 0$ and the outer boundary is insulated.

(i) Use Exercise 3 to show the uniqueness, up to a constant, of any solution to this boundary value problem.

(ii) Find all circularly symmetric solutions $T(r)$ to

$$\nabla^2 T = \frac{\mathrm{d}^2 T}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}T}{\mathrm{d}r} = 1$$

in the annulus.

(iii) For what value of k is there a circularly symmetric solution to this boundary value problem? Interpret this value physically.

5. Let R be the region $x^2/a^2 + y^2/b^2 + z^2/c^2 \leqslant 1$ with boundary ∂R and $a, b, c > 0$. Suppose $u(x, y, z)$ satisfies $\nabla^2 u = -1$ in R and $u = 0$ on ∂R.

(i) Show that the solution u is unique.

(ii) Show that the solution u is a quadratic function of x, y, z and evaluate

$$\iint_{\partial R} \nabla u\cdot \mathrm{d}\mathbf{S}.$$

6. (Optional) A one-dimensional blob of compressible fluid starts at $t = 0$ with uniform density $\rho = 1$ in the interval $1 \leqslant x \leqslant 2$ and moves with velocity given by

$$u(x,t) = 2x^2 t.$$

(i) Find the position $x(t)$ at time t of a fluid particle that starts at $x(0) = a$ where $1 \leqslant a \leqslant 2$.

(ii) Hence show that the density of the fluid is given by

$$\rho(x,t) = \frac{1}{(1 + xt^2)^2}.$$

(iii) Verify that $\rho(x,t)$ satisfies the *continuity equation*

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0.$$

(iv) Use the continuity equation to find a condition on $u(x,t)$ for a fluid to be incompressible.