

MULTIVARIABLE CALCULUS HT19 SHEET 8
Gravity. Gauss' Flux Theorem. Poisson's Equation.

1. Two planets, modelled as particles of equal mass M , are situated at the points $(a, 0, 0)$ and $(-a, 0, 0)$ and remain there. Show that the gravitational potential at the point $(x, 0, 0)$, where $-a < x < a$, is

$$\frac{2GMa}{a^2 - x^2},$$

and find the gravitational field at $(x, 0, 0)$. A space-craft moves along the segment $-a < x < a$ of the x -axis under the gravitational attraction of the planets. Write down the equation of motion for the space-craft and show that $x = 0$ is a point of unstable equilibrium.

At $x = 0$ the space-craft fires its motor briefly so as to acquire a speed $u(> 0)$ and heads towards $x = a$. By using energy considerations show that the time taken to reach $x = a$ is

$$T = \int_0^a \sqrt{\frac{a^2 - x^2}{u^2(a^2 - x^2) + \frac{4GM}{a}x^2}} dx.$$

(You are not required to evaluate the integral.)

2. A homogeneous straight wire of mass M lies along the x -axis from $(-a, 0, 0)$ to $(a, 0, 0)$. Show that the gravitational potential at the point $(x, y, 0)$, where $y \neq 0$, is

$$\frac{GM}{2a} \left[\sinh^{-1} \left(\frac{a-x}{|y|} \right) + \sinh^{-1} \left(\frac{a+x}{|y|} \right) \right]$$

and that the x -component of the gravitational field is

$$\frac{GM}{2a} \left(\frac{1}{d_1} - \frac{1}{d_2} \right),$$

where d_1 and d_2 are the distances from $(x, y, 0)$ to the ends of the wire.

3. Show that the gravitational field at the vertex of a homogeneous circular cone is

$$\frac{12GM}{a^2} \sin^2 \frac{\alpha}{2},$$

where M is the mass of the cone, a is the radius of the base and α is the semi-vertical angle.

4. A hollow spherical shell has internal radius a and external radius b , and is made of material of uniform density ρ . Find the gravitational field and the gravitational potential in the three regions $0 < r < a$, $a < r < b$, $r > b$ by using (1) the Flux Theorem, and (2) Poisson's equation.

5. A solid hemisphere of uniform density ρ occupies the region

$$x^2 + y^2 + z^2 \leq a^2, \quad z \leq 0.$$

Find the gravitational potential due to the hemisphere at the point $(0, 0, s)$ where $s > 0$. A uniform rod of density m per unit length lies on the z -axis between $(0, 0, c)$ and $(0, 0, d)$ where $d > c > 0$. Show that the force exerted on the rod by the hemisphere is

$$\psi(c) - \psi(d)$$

where

$$\psi(\lambda) = \frac{2\pi Gm\rho}{3} \left(\frac{a^3 + \lambda^3 - (a^2 + \lambda^2)^{3/2}}{\lambda} \right).$$