Dynamics: Problem Sheet 1 (of 8)

A first look at forces and dynamics: gravity and projectiles, fluid drag.

1. Consider the following model for jumping vertically. While in contact with the ground your legs provide a constant force F_0. Suppose that in a crouched position you lower your centre of mass by L metres. Thus if x is the height of your centre of mass in metres from the standing position, and m is your mass, the vertical force acting is

$$F(x) \quad = \quad \begin{cases} F_0 - mg & -L < x < 0 \ , \\ -mg & x > 0 \end{cases} .$$

   (a) Suppose that you start at rest in the crouched position ($\dot{x}(0) = 0$, $x(0) = -L$). By solving Newton's second law $m\ddot{x} = F(x)$, show that your vertical velocity at the time your feet leave the ground, *i.e.* when $x = 0$, is

$$v \quad = \quad \sqrt{2L\left(\frac{F_0}{m} - g\right)} \ .$$

   (b) Show that you reach a maximum height at a time

$$t \quad = \quad \sqrt{\frac{2L}{\frac{F_0}{m} - g}} + \frac{\sqrt{2L(\frac{F_0}{m} - g)}}{g} \ ,$$

   and that this height is $x = L\left(\frac{F_0}{mg} - 1\right)$.

2. A cannon at the origin O fires a shell with speed V at an angle α to the horizontal.

   (a) Write down Newton's second law for the shell, with suitable initial conditions. Solve this differential equation to show that the trajectory of the shell is given by

$$x(t) \quad = \quad t\,V\cos\alpha \ , \qquad z(t) \quad = \quad -\frac{1}{2}gt^2 + t\,V\sin\alpha \ .$$

   (b) Obtain the equation for the path of the shell, expressing the height z as a function of x.

   (c) Suppose now that V is fixed but the angle α may be varied. Find the upper boundary curve $z = z(x)$ of the set of points in the (x, z) plane which it is possible to hit with a shell.

3. Consider a particle of mass m moving vertically in a fluid with quadratic drag force Dv^2, where v is its velocity and $D > 0$ is a constant. The particle is also acted on by gravity, with acceleration due to gravity g.

   (a) Consider dropping the particle from rest through the fluid, so that its velocity is $v = \dot{z} \leq 0$, with z measured upwards. Show that the equation of motion may be written as

$$m\dot{v} \quad = \quad -mg + Dv^2 \ .$$

Show that this may be integrated to

$$t = -\int_0^v \frac{\mathrm{d}u}{g - \frac{Du^2}{m}} .$$

By evaluating the integral, hence show that the solution is

$$v(t) = -\sqrt{\frac{mg}{D}} \tanh\left(\sqrt{\frac{Dg}{m}}\, t\right) .$$

What is the terminal velocity?

(b) Now consider projecting the particle upwards through the fluid, starting at $z = 0$ with speed u. Show that the equation of motion may be written as

$$\frac{\mathrm{d}(v^2)}{\mathrm{d}z} = 2\dot{v} = -2g - \frac{2Dv^2}{m} .$$

Regarding this as an equation for $v^2(z)$, by integrating it show that the maximum height reached is

$$z_{\mathrm{max}} = \frac{m}{2D} \log\left(1 + \frac{Du^2}{mg}\right) .$$

What happens as $D \to 0$?

Please send comments and corrections to `sparks@maths.ox.ac.uk`.