Dynamics: Problem Sheet 2 (of 8)

Forced oscillations, examples of motion in two and three dimensions, dimensional analysis.

1. A particle of mass m moves along the x axis with one end attached to a spring of spring constant $k > 0$, and is subjected to an additional force $F_0 \cos \Omega t$.

   (a) Show that the equation of motion is

   $$\ddot{x} + \omega^2 x = A \cos \Omega t ,$$

   where $x = 0$ corresponds to the unstretched position of the spring, $\omega = \sqrt{k/m}$, and $A = F_0/m$.

   (b) Suppose that $x = \dot{x} = 0$ at time $t = 0$. Verify that if $\Omega \neq \omega$ then

   $$x(t) = \frac{A}{\omega^2 - \Omega^2} (\cos \Omega t - \cos \omega t)$$

   satisfies the equation of motion and initial conditions, while when $\Omega = \omega$ then

   $$x(t) = \frac{A}{2\omega} t \sin \omega t$$

   does. What is the qualitative difference between the two solutions?

2. Consider a particle of mass m moving in a plane with position vector $\mathbf{r} = (x, y)$, subject to a force $\mathbf{F} = -k\,\mathbf{r}$, where $k > 0$ is constant.

   (a) Show that the general solution to the equation of motion is

   $$\mathbf{r}(t) = \mathbf{A} \sin \omega t + \mathbf{B} \cos \omega t ,$$

   where $\mathbf{A}$ and $\mathbf{B}$ are constant vectors, and $\omega = \sqrt{k/m}$. (You might find it helpful to write out the vector equation of motion in terms of its components.)

   (b) Show that the solution in part (a) may be rewritten as

   $$\mathbf{r}(t) = \mathbf{a} \sin(\omega t + \phi) + \mathbf{b} \cos(\omega t + \phi) ,$$

   where now $\mathbf{a}$ and $\mathbf{b}$ are constant *orthogonal* vectors, and ϕ is a constant phase.

   (c) Hence show that the trajectory of the particle traces out an ellipse, with centre the origin.

3. Consider a particle of charge q moving in a constant electromagnetic field. Without loss of generality we take the magnetic field $\mathbf{B} = (0, 0, B) \neq 0$ to point along the z axis, while the electric field $\mathbf{E} = (E_1, E_2, E_3)$ is constant, but arbitrary.

   (a) Assuming the particle has mass m, but ignoring gravity, show that Newton's second law implies the coupled ODEs

   $$\begin{aligned} m\ddot{x} &= q\,E_1 + qB\,\dot{y} , \\ m\ddot{y} &= q\,E_2 - qB\,\dot{x} , \\ m\ddot{z} &= q\,E_3 , \end{aligned}$$

   for the position $\mathbf{r} = (x, y, z)$ of the particle.

(b) Verify that

$$
\begin{aligned}
x(t) &= x_0 + \frac{E_2}{B}\,t + R\,\cos(\omega t + \phi)\ , \\
y(t) &= y_0 - \frac{E_1}{B}\,t - R\,\sin(\omega t + \phi)\ , \\
z(t) &= z_0 + u\,t + \frac{q}{2m} E_3\,t^2\ ,
\end{aligned}
$$

solves the equations of motion in part (a), where $\omega = qB/m$ is the cyclotron frequency, (x_0, y_0, z_0) is a constant vector, and u, R and ϕ are also constants.

[*Optional*: For a more challenging version of this question, rather than verifying the solution, instead *derive* it, hence showing it is the general solution.]

4. (a) Find the dimensions of kinetic energy ($\frac{1}{2}m|\dot{\mathbf{r}}|^2$) and linear momentum ($\mathbf{p} = m\dot{\mathbf{r}}$) in terms of the fundamental dimensions L, M and T.

   (b) (The atomic bomb): An essentially instantaneous release of an amount of energy E from a very small volume creates a rapidly expanding high pressure fireball, bounded by a very strong thin spherical shock wave across which the pressure drops abruptly to atmospheric. The pressure inside the fireball is so great that the ambient atmospheric pressure is negligible by comparison, and the only property of the air that determines the radius $r(t)$ of the fireball is its density ρ (the mass of air per unit volume). Show dimensionally, by identifying the only possible combination of E, t and ρ, that

$$
r(t) \quad \sim \quad E^{1/5}\,t^{2/5}\,\rho^{-1/5}\ .
$$

   This result, due to G. I. Taylor, can be used to deduce E from observations of $r(t)$. Taylor's publication of this apparently caused considerable embarrassment in US military scientific circles, where it was regarded as top secret.

Please send comments and corrections to `sparks@maths.ox.ac.uk`.