Dynamics: Problem Sheet 4 (of 8)

Coupled oscillations, energy and angular momentum, circular motion.

1. Consider the following system of coupled second order ODEs for $x(t)$, $y(t)$:

$$\ddot{x} = 1 + \sin y - e^{3x} ,$$
$$\ddot{y} = e^{x-3y} - 1 .$$

   (a) Show that $(x, y) = (0, 0)$ is an equilibrium configuration, and that the linearized equations of motion about this point are

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = M \begin{pmatrix} x \\ y \end{pmatrix} , \qquad \text{where } M = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix} .$$

   (b) By determining the eigenvalues and eigenvectors of M, hence show that the *normal mode solutions* to the equations in part (a) are

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = A \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\sqrt{2}t + \phi) , \qquad \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = B \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(2t + \psi) ,$$

   where A, B, ϕ and ψ are constants.

2. A particle of mass m moves in $\mathbb{R}^3$ under the influence of a force $\mathbf{F} = -k\,\mathbf{r}$, where $\mathbf{r}$ is the position vector of the particle and $k > 0$ is constant.

   (a) Explain why $\mathbf{F}$ is both a *conservative* force, and a *central* force, where a choice of potential energy function is $V(\mathbf{r}) = \frac{1}{2}k|\mathbf{r}|^2$. Hence deduce that the particle moves in a plane through the origin.

   (b) Taking the plane of motion to be the (x, y) plane, the solution to the equation of motion may be written as

$$\mathbf{r}(t) = a \sin(\omega t + \phi)\,\mathbf{i} + b \cos(\omega t + \phi)\,\mathbf{j} ,$$

   where $\omega = \sqrt{k/m}$, and a, b and ϕ are constant. (This solution was found on Problem Sheet 2, question 2.) Assuming this solution, compute the total energy E and total angular momentum $\mathbf{L}$ about the origin, thus confirming that both are indeed constant. Show in particular that the *specific angular momentum* $|\mathbf{L}|/m = 2A/T$, where A is the area of the ellipse traced out by the solution, and T is the period of the solution.

3. At a given instant of time, a particle of mass m has position vector $\mathbf{r}$, measured from the origin O of an inertial frame, and velocity $\mathbf{v}$. Let $\mathcal{L}$ be the straight line through $\mathbf{r}$ with tangent vector $\mathbf{v}$. Show that the angular momentum $\mathbf{L}_O$ of the particle about O has magnitude $|\mathbf{L}_O| = d|\mathbf{p}|$, where d is the perpendicular distance between O and $\mathcal{L}$, and $\mathbf{p}$ is the (linear) momentum of the particle. When is $\mathbf{L}_O = \mathbf{0}$?

4. A point particle moves on a circle of radius l in the (z, x) plane, centred on the origin.

   (a) By introducing polar coordinates $(z, x) = (-r\cos\theta, r\sin\theta)$, show that the particle has acceleration

$$\ddot{\mathbf{r}} = -l\dot{\theta}^2\,\mathbf{e}_r + l\ddot{\theta}\,\mathbf{e}_\theta ,$$

   where $\mathbf{e}_r = -\cos\theta\,\mathbf{k} + \sin\theta\,\mathbf{i}$, $\mathbf{e}_\theta = \sin\theta\,\mathbf{k} + \cos\theta\,\mathbf{i}$.

(b) Suppose that the particle has mass m, and that the acceleration in part (a) arises from Newton's second law with a total force

$$\mathbf{F} \quad = \quad -mg\,\mathbf{k} + \mathbf{T} \; .$$

Show that

$$\mathbf{T} \cdot \mathbf{e}_r \quad = \quad -ml\dot{\theta}^2 - mg\cos\theta \; .$$

(c) Consider a ball of mass m fixed to one end of a light (negligible mass) string of length l. Holding the other end of the string at a fixed point in space, you swing the ball and string such that the string remains taut (straight), and the motion lies in a vertical plane. Denoting the angular velocity of the ball by $\omega = \dot{\theta}$, explain why the string never becomes slack if $\omega > \sqrt{g/l}$ throughout the motion. What happens if $\omega \leq \sqrt{g/l}$?

Please send comments and corrections to `sparks@maths.ox.ac.uk`.