Dynamics: Problem Sheet 5 (of 8)

Constrained motion, (optional) radial Kepler problem.

1. A particle moves on the inside surface of a smooth cone with its axis vertical, defined by the equation $z = r$ in cylindrical polar coordinates (r, θ, z). Initially the particle is at height $z = a$, and its velocity is horizontal, speed v, in the $\mathbf{e}_\theta$ direction. Starting from Newton's second law show that $r^2\dot{\theta}$ is constant. Explain why the total energy is conserved, and deduce that

$$\dot{z}^2 + \frac{1}{2}\frac{a^2v^2}{z^2} + gz \quad = \quad \frac{1}{2}v^2 + ga \ .$$

   Hence show that the particle remains at all times between two heights, which should be determined.

2. A particle of mass m, moving under gravity, is disturbed from rest at the highest point on the outside of a smooth sphere of radius a.

   (a) Explain why the particle subsequently moves on a great circle.

   (b) By introducing plane polar coordinates in the vertical plane of this circle (or otherwise), show that

$$\ddot{\theta} \quad = \quad \frac{g}{a}\sin\theta \ , \qquad N \quad = \quad mg\cos\theta - ma\dot{\theta}^2 \ .$$

   Here $\theta(t)$ denotes the angle between the *upward* vertical axis of the sphere and the straight line from the particle to the centre of the sphere (the usual polar angle for a sphere), and N is the magnitude of the normal reaction.

   (c) Show that the normal reaction is given by

$$N \quad = \quad mg\left(\frac{3z}{a} - 2\right) \ ,$$

   where z is the height of the particle above the centre of the sphere. At what height does the particle lose contact with the sphere?

3. A bead of mass m is free to slide on a smooth wire that is made to rotate at constant angular velocity ω about the vertical axis through a fixed point O on the wire. The wire is bent into the shape of a parabola, $z = r^2/2a$, where z is measured vertically upwards from O, and r is the horizontal distance from O.

   (a) Show that if $z(t)$ and $r(t)$ are the vertical and horizontal distances of the bead from O, then

$$m\left[(\ddot{r} - r\omega^2)\mathbf{e}_r + 2\omega\dot{r}\,\mathbf{e}_\theta + \ddot{z}\,\mathbf{e}_z\right] \quad = \quad -mg\,\mathbf{e}_z + \mathbf{N} \ ,$$

   where $\mathbf{N}$ is the normal reaction.

   (b) Hence deduce that

$$(a^2 + r^2)\ddot{r} + r\dot{r}^2 \quad = \quad (a^2\omega^2 - ga)r \qquad\qquad (*) \ .$$

(c) Show that $r=0$ is an equilibrium point. The *linearized equation of motion* about $r=0$ is

$$a^2\ddot{\xi} \quad = \quad (a^2\omega^2 - ga)\xi \ ,$$

where we have written $r=\xi$, and kept only the linear terms in $\xi, \dot{\xi}$ in equation $(*)$, in a Taylor expansion around $\xi = 0$. Discuss the stability of the equilibrium point.

4. (*Optional*) A particle of mass m is released from rest at a very large height $z = z_0$ above the Earth. The Newtonian gravitational potential energy of the particle is

$$V(z) \quad = \quad -\frac{G_N M\, m}{z} \ ,$$

where M is the mass of the Earth, and G_N is Newton's gravitational constant.

(a) Using conservation of energy show that the trajectory $z(t)$ satisfies

$$\sqrt{2G_N M}\, t \quad = \quad -\int_{z_0}^{z} \left(\frac{1}{s} - \frac{1}{z_0}\right)^{-1/2} \mathrm{d}s \ .$$

(b) Using the substitution $s = z_0 \sin^2\theta$, show $z(t)$ satisfies the unlikely looking equation

$$\frac{\pi}{2} - \sin^{-1}\left(\sqrt{\frac{z(t)}{z_0}}\right) + \frac{1}{2}\sin\left[2\sin^{-1}\left(\sqrt{\frac{z(t)}{z_0}}\right)\right] \quad = \quad \sqrt{\frac{2G_N M}{z_0^3}}\, t \ .$$

This is a *radial Kepler trajectory* (*c.f.* section 6.2 of the lecture notes).

Please send comments and corrections to `sparks@maths.ox.ac.uk`.