**Final Honour School of Mathematics Part A**

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# Algebra 2
# Revised for Rings & Modules course in AY 2021–2022

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**June 2020**

**Do not turn this page until you are told that you may do so**

1. Suppose that $R$ is a commutative ring.
   (a) [5 marks] Show that the following are equivalent:
      (i) $R$ is an integral domain;
      (ii) $R[X]$ is an integral domain;
      (iii) if $p, q \in R[X]^*$ then $pq \in R[X]^*$ and $\deg pq = \deg p + \deg q$.
   (b) [5 marks] Let $R$ be an integral domain which contains a field $K$ such that $R$ is a finite dimensional vector space over $K$. Prove that $R$ itself is a field.
   (c) [2 marks] Show that the finite dimension requirement in part (b) cannot be dropped.
   (d) [7 marks] State and prove the *Tower Law*.
   (e) [2 marks] Give an example of a field extension of $\mathbb{C}$ that is not equal to $\mathbb{C}$.
   (f) [4 marks] Show that your example is not an extension of finite degree.

2. (a) [2 marks] Define what it means for a ring to be a *Euclidean domain*.

   (b) [5 marks] Show that if $\mathbb{F}$ is a field then $\mathbb{F}[X]$ is a Euclidean domain.

   (c) [7 marks] Show that $\mathbb{F}[X, Y]$ is not a Euclidean domain.

   (d) [6 marks] Show that if $R$ is an infinite PID with finitely many units then it has infinitely many maximal ideals.

   [*You may assume that every proper ideal is contained in a maximal ideal.*]

   (e) [5 marks] The set $R := \{q/r : q, r \in \mathbb{Z} \text{ and } r \text{ is odd}\}$ is an infinite subring of $\mathbb{Q}$. Show, with proof, that it is a PID with only one proper maximal ideal, namely $\langle 2 \rangle$.

3. Suppose that $R$ is a ring which need not be commutative.

(a) [3 marks] Explain how the quotient of the $R$-module $R$ by a submodule has the structure of an $R$-module, and how the direct sum of two $R$-modules has the structure of a $R$-module.

(b) [7 marks] Suppose that $J_1$ and $J_2$ are submodules of the $R$-module $R$ with $J_1 + J_2 = R$. Show that the map

$$R \to (R/J_1) \oplus (R/J_2); \quad r \mapsto (r + J_1, r + J_2)$$

is a surjective $R$-linear map.

For the remainder of the question let $\mathbb{F}$ be a field, $V$ a vector space over $\mathbb{F}$, and $R = \mathrm{End}_{\mathbb{F}}(V)$.

(c) [4 marks] Explain briefly how and why $R$ is a ring.

(d) [1 mark] Suppose that $V$ is 2-dimensional. Show that there are vectors $e_1$, $e_2$, $e_3$ such that any two of these is a basis for $V$.

For the remainder of the question assume that $e_1$, $e_2$, and $e_3$ are three vectors such that any two are a basis for $V$, but they are not necessarily those found in part (d).

(e) [4 marks] Show that $J_i := \{T \in R : T(e_i) = 0\}$ is a submodule of the $R$-module $R$ and also a 2-dimensional $\mathbb{F}$-vector space for each $1 \leqslant i \leqslant 3$.

(f) [4 marks] Show that $J_i + J_j = R$ for all $1 \leqslant i < j \leqslant 3$.

(g) [2 marks] Show that the map

$$R \to (R/J_1) \oplus (R/J_2) \oplus (R/J_3); \quad r \mapsto (r + J_1, r + J_2, r + J_3)$$

is not surjective.