Final Honour School of Mathematics Part A

Algebra 2
Revised for Rings & Modules course in AY 2021–2022

June 2021

Do not turn this page until you are told that you may do so

1. (a) [12 marks]

   (i) Suppose that R is a PID, $f(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0 \in R[X]$, and $p \in R$ is prime such that $p|a_i$ for all $0 \leqslant i < n$ but $p^2 \not| a_0$. Show that f is irreducible in $R[X]$.

   (ii) Show that $Y^2 + X$ and $Y^2 + 2Y + 1 + X$ are both irreducible in $\mathbb{R}[X, Y]$.

   (b) [13 marks] Write T for the subring of polynomials in $\mathbb{R}[X]$ for which the coefficient of X is 0.

   (i) Show that if $p \in T$ then there are unique reals a, b, and c such that $p(X) \in a + bX^2 + cX^4 + \langle X^3 \rangle$.

   (ii) Show that

$$T/\langle X^3\rangle \to \mathbb{R}[Y]/\langle Y^3\rangle; a + bX^2 + cX^4 + \langle X^3\rangle \mapsto a + bY + cY^2 + \langle Y^3\rangle$$

   is a ring isomorphism.

   We say that a property P of rings is invariant under isomorphism if whenever R is a ring with property P, and S is a ring isomorphic to R, then S also has property P.

   (iii) Is there a property P of rings invariant under isomorphism such that $x \in R$ is irreducible if and only if $R/\langle x\rangle$ has property P? Justify your answer.

   (iv) State a property P of rings invariant under isomorphism such that $x \in R$ is prime if and only if $R/\langle x\rangle$ has property P.

[*Throughout this question you may assume standard results from the course provided they are clearly stated.*]

2. Suppose that R is a commutative ring and M and N are R-modules with N finitely generated.

   (a) [3 marks] Show that if $\phi : N \to M$ is a surjective R-linear map then M is finitely generated.

   (b) [4 marks] Show that if N is a submodule of M, and M/N is finitely generated then M is finitely generated.

   For the remainder of the question, suppose that $\phi : R^m \to M$ and $\psi : R^n \to M$ are both surjective R-linear maps.

   (c) [4 marks] Show that there is an R-linear map q such that $\phi \circ q = \psi$.

   (d) [10 marks] Write N for the R-module $\{q(x) : x \in \ker \psi\}$, and show that the map
   $$\Psi : \ker \phi / N \to R^m / \operatorname{Im} q; y + N \mapsto y + \operatorname{Im} q$$
   is a well-defined R-linear bijection.

   (e) [4 marks] Hence show that if $\ker \psi$ is finitely generated then so is $\ker \phi$.

3. (a) [5 marks] Suppose that G is a commutative group with generators a, b, and c of orders 4, 12, and 24 respectively, and satisfying the relation $3a + 2b = 0$. Find $n \in \mathbb{N}_0$ and $d_1 | \dots | d_n$ such that $G \cong \mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_n}$.

   Suppose that d and h are coprime natural numbers.

   (b) [4 marks] Show that $d\mathbb{Z} \cap h\mathbb{Z} = dh\mathbb{Z}$.
   [*You may assume $\mathbb{Z}$ is a PID.*]

   (c) [7 marks] Hence show that $\mathbb{Z}_d \oplus \mathbb{Z}_h \cong \mathbb{Z}_{dh}$.

   We say a natural number N is square-free if $m^2 | N$ for $m \in \mathbb{N}^*$ implies $m = 1$.

   (d) [6 marks] Use the previous part to show that if G is a direct sum of cyclic groups and has square-free size then G is cyclic.

   (e) [3 marks] Use the structure theorem for finite commutative groups to show that if $\mathbb{F}$ is a finite field then $U(\mathbb{F})$ is cyclic.