Analysis II: Continuity and Differentiability   Sheet 1   HT 2019

1. Suppose that f, g are real-valued functions defined on some interval (a,b) containing the point x_0, and that $\lim_{x\to x_0} f(x) = k$ and $\lim_{x\to x_0} g(x) = l$.

(a) If $k \neq 0$, prove that there is a positive number $\delta > 0$ such that $|f(x)| \geq \frac{1}{2}k$ whenever $0 < |x - x_0| < \delta$.

(b) Prove that: $\lim_{x\to x_0} (f(x) + g(x)) = k + l$; $\lim_{x\to x_0} f(x)g(x) = kl$ and $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{k}{l}$ provided $l \neq 0$.

(c) Prove that if $f(x) < g(x)$ for all $x \in (x_0 - \delta, x_0 + \delta)$ (for some $\delta > 0$) then $k \leq l$. In this case is it true that $k < l$?

[*Hint: mimic what was done for limits of sequences.*]

2. Prove carefully that if $f(x) \to y_0$ as $x \to x_0$, $g(y) \to l$ as $y \to y_0$, and if $g(y_0) = l$, then $g(f(x)) \to l$ as $x \to x_0$.

3. (a) Find a function $f : \mathbb{R} \to \mathbb{R}$ which is discontinuous at the points of set $\{1/n : n \in \mathbb{Z}_+\} \cup \{0\}$ but is continuous everywhere else.

(b) Find a function $g : \mathbb{R} \to \mathbb{R}$ which is discontinuous at the points of set $\{1/n : n \in \mathbb{Z}_+\}$ but is continuous everywhere else.

4. (a) Prove that if f is continuous at x_0, then $|f|$ is continuous at x_0.

(b) For x, $y \in \mathbb{R}$, show that $\max\{x,y\} = \frac{1}{2}(x + y + |x - y|)$ and that $\min\{x,y\} = \frac{1}{2}(x + y - |x - y|)$.

(c) Prove that if f and g are continuous at x_0, then $\max\{f,g\}$ and $\min\{f,g\}$ are continuous at x_0.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying

$$f(x+y) = f(x) + f(y) \qquad \text{for any } x, y \in \mathbb{R}\ .$$

Prove that $f(x) = cx$ (for every $x \in \mathbb{R}$) for some constant c.

[Hint: First show that $f(0) = 0$, $f(-x) = -f(x)$ for every x, and use induction to show that $f(nx) = nf(x)$, then show that $f(rx) = rf(x)$ for any rational number $r = n/m$.]