Analysis II: Continuity and Differentiability Sheet 2 HT 2019

1. If $f:(0,1]\to\mathbb{R}$ defined by $f(x)=0$ if x is irrational, and $f(x)=\frac{1}{p+q}$ if $x=p/q\in(0,1]$ where $(p,q)=1$ (i.e. p and q are co-prime), show that f is continuous at any irrational point in $[0,1]$ and discontinuous at rational numbers in $(0,1]$.

2. Give examples of

(a) a continuous function $f:[0,1)\to\mathbb{R}$ which is not bounded;

(b) a continuous function $g:[0,1)\to\mathbb{R}$ which is bounded but does not attain one of its bounds;

(c) a continuous function $h:[0,1)\to\mathbb{R}$ which is bounded but does not attain either of its bounds.

[*Of course, if you manage (c) you'll have done (b) as well.*]

3. Assuming the theorem that a continuous real-valued function on a *closed bounded* interval is bounded and attains its bounds, prove that if $f:\mathbb{R}\to\mathbb{R}$ is continuous and $f(x)\to+\infty$ as $x\to\pm\infty$ then there exists some $x_0\in\mathbb{R}$ such that $f(x)\geq f(x_0)$ for all $x\in\mathbb{R}$.

4. (a) Show that every polynomial of odd degree with real coefficients has at least one real root.

(b) Let f be a continuous real-valued function on $[a,b]$. Suppose that B is a real number such that $f(x)\neq B$ for every $x\in[a,b]$, show that $f(x)<B$ for all $x\in[a,b]$ or $f(x)>B$ for all $x\in[a,b]$.

5. How many continuous functions $f:\mathbb{R}\to\mathbb{R}$ are there such that

$$(f(x))^2=x^2$$

for all $x\in\mathbb{R}$? (Justify your answer.)